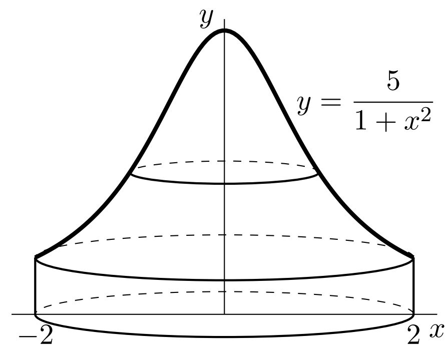
3. [12 points] A new top secret weather balloon has the ability to make it rain orange soda. The base of the balloon is a solid cylinder with a radius of 2 meters and a height of 1 meter. Above that is a solid obtained by taking the portion of the function $y = \dfrac{5}{1+x^2}$ for $0 \le x \le 2$ and rotating it around the y-axis (where x and y are measured in meters). The balloon is made of a light metal which has a constant density δ kg/m^3. The balloon is pictured on the right. (The picture is not to scale.)



a. [4 points] Write down an expression in terms of y (but not x) that approximates the volume, in cubic meters, of a horizontal slice of the weather balloon of thickness Δy at a height y meters above the ground where $1 < y < 5$.

Solution: Solving the equation $y = \frac{5}{1+x^2}$ for x gives the radius of a circular cross section at height y. The volume of the slice is then approximately $\pi\left(\frac{5}{y} - 1\right) \cdot \Delta y$ cubic meters

b. [4 points] Write down an expression involving one or more integrals which gives the total mass of the weather balloon in kilograms. Do **not** evaluate any integrals in this expression.

Solution: We integrate the solution from part (a) from 1 to 5 replacing Δy with dy and then add 4π (the volume of the cylindrical base) to get the total volume of the balloon. Finally we multiply this entire expression by the constant density δ to obtain a total mass of

$$4\pi\delta + \int_1^5 \delta\pi\left(\frac{5}{y} - 1\right)\, dy \quad \text{kilograms.}$$

c. [4 points] Write down an expression involving one or more integrals which gives the y–coordinate of the center of mass of the weather balloon. Do **not** evaluate any integrals in this expression.

Solution: We rewrite the first term in (b) as $\displaystyle\int_0^1 4\pi\, dy$ and include a factor of y in each integral and then divide by the total mass to find that the y–coordinate of the center of mass is

$$\frac{\displaystyle\int_0^1 y\cdot 4\pi\delta\, dy + \int_1^5 y\cdot\delta\pi\left(\frac{5}{y} - 1\right)\, dy}{\displaystyle 4\pi\delta + \int_1^5 \delta\pi\left(\frac{5}{y} - 1\right)\, dy}.$$