1. [14 points] Let $f(x)$ be a twice-differentiable function. Use the table to compute the following expressions. Show your work.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	1	2	4	11	1	3	5	4	2	3
$f'(x)$	2	3	7	4	-5	2	1	-2	-3	1

a. [3 points] $\displaystyle\int_1^8 \frac{f'(\sqrt[3]{x})}{\sqrt[3]{x^2}}\,dx$

Solution: Use u-substitution with $u = \sqrt[3]{x}$ to get

$$u = \sqrt[3]{x}$$
$$du = \frac{1}{3}\frac{1}{\sqrt[3]{x^2}}dx.$$

Then the antiderivative is

$$\int \frac{f'(\sqrt[3]{x})}{\sqrt[3]{x^2}}\,dx = 3\int f'(u)\,du = 3f(\sqrt[3]{x}).$$

Hence we have

$$\int_1^8 \frac{f'(\sqrt[3]{x})}{\sqrt[3]{x^2}}\,dx = 3(f(2) - f(1)) = 6.$$

Answer: a. **6**

b. [3 points] $\displaystyle\int_7^9 \frac{12f'(x)}{(f(x))^2}\,dx$

Solution: Use u-substitution with $u = f(x)$ to get

$$u = f(x)$$
$$du = f'(x)dx.$$

Then the antiderivative is

$$\int \frac{12f'(x)}{(f(x))^2}\,dx = 12\int \frac{1}{u^2}\,du = -\frac{12}{f(x)}.$$

Hence we have

$$\int_7^9 \frac{12f'(x)}{(f(x))^2}\,dx = -\frac{12}{f(9)} + \frac{12}{f(7)} = -\frac{12}{3} + \frac{12}{4} = -1.$$

Answer: b. -1

c. [3 points] $\int_0^3 x f''(x)\, dx$

Solution: Use integration by parts with $u = x$ and $dv = f''(x)dx$ to get

$$\begin{aligned} u &= x & v &= f'(x) \\ du &= dx & dv &= f''(x)dx. \end{aligned}$$

Then the antiderivative is

$$\int x f''(x)\, dx = x f'(x) - \int f'(x)\, dx = x f'(x) - f(x).$$

$$\int_0^3 x f''(x)\, dx = (3f'(3) - f(3)) - (0f'(0) - f(0)) = (3 \cdot 4 - 11) - (0 \cdot 2 - 1) = 2.$$

Answer: c. **2**

d. [5 points] The average value of $\dfrac{2f'(x)}{(f(x))^2 + f(x)}$ on $[4, 6]$.

Solution: By definition, this average is given by the integral

$$\frac{1}{6-4}\int_4^6 \frac{2f'(x)}{(f(x))^2 + f(x)}\, dx = \int_4^6 \frac{f'(x)}{(f(x))^2 + f(x)}\, dx.$$

To compute this antiderivative, first use the u-substitution $u = f(x)$ to get

$$\int \frac{f'(x)}{(f(x))^2 + f(x)}\, dx = \int \frac{1}{u^2 + u}\, du.$$

Using the method of partial fractions we find the identity

$$\frac{1}{u^2 + u} = \frac{1}{u} - \frac{1}{u+1}.$$

Thus

$$\int \frac{1}{u^2 + u}\, du = \ln|u| - \ln|u+1|,$$

and changing variables back to x leads to

$$\int \frac{f'(x)}{(f(x))^2 + f(x)}\, dx = \ln|f(x)| - \ln|f(x) + 1|.$$

Therfore,

$$\begin{aligned} \int_4^6 \frac{f'(x)}{(f(x))^2 + f(x)}\, dx &= (\ln|f(6)| - \ln|f(6)+1|) - (\ln|f(4)| - \ln|f(4)+1|) \\ &= \ln(5) - \ln(6) + \ln(2) \\ &\approx 0.510825 \end{aligned}$$

Answer: d. $\ln(5) - \ln(6) + \ln(2) \approx 0.510825$