5. [9 points] Tammy Toppel is directing a performance art piece at the community center. She fills a large cone with sand and cuts a small hole in the bottom. Gerd Hömf was hired from a temp agency to stand behind the scenes and steadily lift the cone with an elaborate pulley system, letting the sand slowly spill onto the stage.

a. [2 points] The filled cone starts with a total mass of 40 kilograms and spills sand at a constant rate of 1/2 a kilogram per second once it is lifted. Tammy wants Gerd to lift the cone at a constant rate of r meters per second. Find a formula for the mass $M(h)$, in kilograms, of the cone when it is h meters above the stage.

Solution: The relation between the height h of the cone and time t is $h = rt$, where r is the rate at which Gerd lifts. So $t = h/r$. The mass as a function of time is given by $40 - \frac{1}{2}t$, hence

$$M(h) = 40 - \frac{h}{2r}.$$

b. [4 points] Gerd lifts the cone until it reaches a height of 20 meters above the stage. Write an integral which represents the work (measured in Joules) done by Gerd while lifting the cone. The integral may include the rate r at which Gerd lifts and g the acceleration (in m/s^2) due to gravity.

Solution: The work done by Gerd lifting the cone to a height of h is given by

$$\int_0^{20} gM(h)\,dh = \int_0^{20} g\left(40 - \frac{h}{2r}\right)\,dh$$

c. [3 points] There's one catch: Gerd's contract strictly prohibits him from exerting more than $500g$ Joules of work, where g is the acceleration due to gravity. At what rate r (in m/s) should Tammy ask Gerd to lift in order to not violate his contract and to get the cone lifted as quickly as possible?

Solution: We set the integral from the previous part equal to $500g$ and solve for r.

$$\begin{aligned} 500g &= \int_0^{20} g\left(40 - \frac{h}{2r}\right)\,dh \\ &= \left(40gh - \frac{h^2}{4r}\right)\Bigg|_0^{20} \\ &= 800g - \frac{400}{4r} - 0 \\ &= g\left(800 - \frac{100}{r}\right). \end{aligned}$$

Dividing both sides by g and multiplying by r we have

$$500r = 800r - 100 \Longrightarrow r = \frac{1}{3}.$$

Answer: $r =$ **1/3**