8. [14 points] Let $g(x)$ be a differentiable function with domain $(-1, 10)$ where some values of $g(x)$ and $g'(x)$ are given in the table below. Assume that all local extrema and critical points of $g(x)$ occur at points given in the table.

x	0	1	2	3	4	5	6	7	8
$g(x)$	2.0	3.3	5.7	6.8	6.0	4.3	2.4	0.2	-4.9
$g'(x)$	2.8	2.5	2.0	0.0	-1.4	-1.9	-1.6	-3.0	-8.1

a. [3 points] Estimate $\int_0^8 g(x)\,dx$ using RIGHT(4). Write out each term in your sum.

Solution: With 4 rectangles the width of each is $\Delta x = \frac{8-0}{4} = 2$. Then

$$\begin{aligned}\text{RIGHT}(4) &= g(2)\Delta x + g(4)\Delta x + g(6)\Delta x + g(8)\Delta x \\ &= (5.7 + 6.0 + 2.4 - 4.9) \cdot 2 \\ &= 18.4\end{aligned}$$

Answer: **18.4**

b. [4 points] Approximate the area of the region between $g(x)$ and the function $f(x) = x + 2$ for $0 \le x \le 4$, using MID(n) to estimate any integrals you use. Use the greatest number of subintervals possible, and write out each term in your sums.

Solution: The function $g(x)$ is concave down on $[0, 4]$, so $g(x)$ is greater than or equal to the linear function $f(x)$ on this interval. The integral to compute this area is

$$\int_0^4 g(x) - f(x)\,dx = \int_0^4 g(x)\,dx - \int_0^4 x + 2\,dx.$$

Since $f(x)$ is linear, we get the same answer whether we use MID to approximate $\int_0^4 g(x) - f(x)\,dx$ or just $\int_0^4 g(x)\,dx$ and compute $\int_0^4 f(x)\,dx$ exactly. In either case, we can use at most 2 subintervals and $\Delta x = 2$.
If we compute MID(2) for $\int_0^4 g(x) - f(x)\,dx$, we get

$$\begin{aligned}\text{MID}(2) &= (g(1) - f(1))\Delta x + (g(3) - f(3))\Delta x \\ &= ((3.3 - 3) + (6.8 - 5))2 \\ &= (.3 + 1.8)2 \\ &= 4.2.\end{aligned}$$

If we compute $\int_0^4 f(x)\,dx = 16$ and then compute MID(2) for $\int_0^4 g(x)\,dx$ we get

$$\begin{aligned}\text{MID}(2) &= g(1)\Delta x + g(3)\Delta x \\ &= (3.3 + 6.8)2 \\ &= 20.2.\end{aligned}$$

Then we get $20.2 - 16 = 4.2$ for the total area.

Answer: **4.2**

c. [3 points] Is your answer to **b.** an overestimate, an underestimate, or is there not enough information to tell? Briefly justify your answer.

Solution: Since we are only given a table and not told that the concavity does not change between points, we technically **do not have enough information** to answer this question.
Had it been the case that $g'(x)$ has no critical points aside from those in the table, it would follow that $g(x)$ is concave down, because $g'(x)$ would be decreasing on the given interval. Since $f(x)$ is linear, the concavity of $g(x) - f(x)$ would also be concave down. In that case, MID(2) would be an **overestimate.**
Credit was awarded for both of these answers.

d. [4 points] Write an integral giving the arc length of $y = g(x)$ between $x = 2$ and $x = 8$. Estimate this integral using TRAP(2). Write out each term in your sum.

Answer: Integral: $\int_2^8 \sqrt{1 + g'(x)^2}\, dx$

Solution: The arc length is given by the integral

$$\text{Arc length} = \int_2^8 \sqrt{1 + g'(x)^2}\, dx.$$

The width of our trapezoids is $\Delta x = \frac{8-2}{2} = 3$.
If we compute the areas of the trapezoids directly we get

$$\begin{aligned}\text{TRAP}(2) &= \left(\frac{\sqrt{1+g'(2)^2} + \sqrt{1+g'(5)^2}}{2}\right)\Delta x + \left(\frac{\sqrt{1+g'(5)^2} + \sqrt{1+g'(8)^2}}{2}\right)\Delta x\\ &\approx (2.1915795 + 5.1542930)3\\ &\approx 22.0376175.\end{aligned}$$

If we compute LEFT(2) and RIGHT(2) first and then take an average we get

$$\begin{aligned}\text{LEFT}(2) &= \sqrt{1+g'(2)^2}\Delta x + \sqrt{1+g'(5)^2}\Delta x\\ &\approx (4.3831590)3\\ &\approx 13.1494770.\end{aligned}$$

$$\begin{aligned}\text{RIGHT}(2) &= \sqrt{1+g'(5)^2}\Delta x + \sqrt{1+g'(8)^2}\Delta x\\ &\approx (10.3085860)3\\ &\approx 30.9257580.\end{aligned}$$

Then

$$\text{TRAP}(2) = \frac{1}{2}(\text{LEFT}(2) + \text{RIGHT}(2)) \approx 22.0376175.$$

Answer: TRAP(2)= **22.0376175**