8. [12 points] Consider the region \( \mathcal{R} \) bounded by the curve \( x^2 + 3y = 4 \) and the \( x \)-axis.

\[ \text{a. [4 points] Write an expression involving one or more integrals that gives the perimeter, in cm, of } \mathcal{R}. \text{ You do not need to evaluate the integral.} \]

**Solution:** The parabola intersects the \( x \)-axis at \( x = \pm 2 \). Moreover, we find
\[ y = \frac{1}{3}(4 - x^2) \text{ so } \frac{dy}{dx} = \frac{2}{3}x. \] We plug this into the arc length formula and add the length along the bottom.

**Answer:**
\[ \int_{-2}^{2} \sqrt{1 + \frac{4}{9}x^2} \, dx + 4 \]

\[ \text{b. [4 points] Write an expression involving one or more integrals that gives the volume, in cm}^3, \text{ of the solid formed by rotating } \mathcal{R} \text{ about the line } x = -4. \]

**Solution:** Using shell method:
\[ \int_{-2}^{2} 2\pi(x - (-4)) \frac{1}{3}(4 - x^2) \, dx \text{ cm}^3 = \frac{2\pi}{3} \int_{-2}^{2} (x + 4)(4 - x^2) \, dx \text{ cm}^3. \]

Using washer method:
\[ \int_{0}^{4/3} \pi \left( \sqrt{4 - 3y - (-4)} \right)^2 - \pi \left( -\sqrt{4 - 3y - (-4)} \right)^2 \, dy \]

**Answer:** (see above)

\[ \text{c. [4 points] Write, but do not evaluate, an expression involving one or more integrals that}
\text{ gives the mass, in grams, of a thin plate in the shape of the region } \mathcal{R} \text{ that has mass density}
\text{ given by } \delta(x) = x + 2 \text{ g/cm}^2. \]

**Answer:**
\[ \int_{-2}^{2} \frac{1}{3}(4 - x^2)(x + 2) \, dx \]