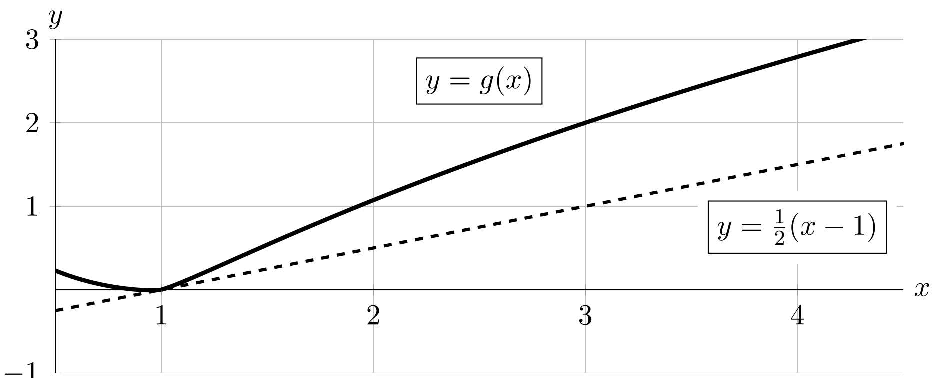
10. [7 points] Consider functions f and g that satisfy all of the following:

- $f(x)$ is defined, positive, and continuous for all $x > 1$.
- $\lim\limits_{x\to 1^+} f(x) = \infty$ (so $f(x)$ has a vertical asymptote at $x = 1$).
- $g(x)$ is defined and differentiable for all real numbers x, and $g'(x)$ is continuous.
- $\dfrac{d}{dx}\left(\dfrac{g(x)}{\ln x}\right) = f(x)$ for all $x > 1$.
- The tangent line to $g(x)$ at $x = 1$ is given by the equation $y = \dfrac{1}{2}(x-1)$. Graphs of $g(x)$ (solid) and this tangent line (dashed) are shown below.



Determine whether the integral $\displaystyle\int_1^3 f(x)\,dx$ converges or diverges.

- If the integral converges, circle "Converges", find its exact value, and write the exact value on the answer blank provided.
- If the integral diverges, circle "Diverges" and carefully justify your answer.

Show every step of your work carefully, and make sure that you use correct notation.

Solution: Since $f(x)$ has a vertical asymptote at $x = 1$, we write

$$\begin{aligned}
\int_1^3 f(x)\,dx &= \lim_{a\to 1^+}\int_a^3 f(x)\,dx \\
&= \lim_{a\to 1^+} \left.\frac{g(x)}{\ln x}\right|_a^3 \\
&= \lim_{a\to 1^+}\left(\frac{g(3)}{\ln 3} - \frac{g(a)}{\ln a}\right) \\
&= \frac{2}{\ln 3} - \lim_{a\to 1^+}\frac{g(a)}{\ln a} \\
&= \frac{2}{\ln 3} - \lim_{a\to 1^+}\frac{g'(a)}{1/a} \quad \text{where we applied l'Hopital's Rule} \\
&= \frac{2}{\ln 3} - \frac{1/2}{1}
\end{aligned}$$

So this improper integral converges.

Circle one:

$\boxed{\displaystyle\int_1^3 f(x)\,dx \textbf{ converges to } \underline{\quad \frac{2}{\ln 3} - \frac{1}{2} \quad}}$ or $\displaystyle\int_1^3 f(x)\,dx$ **diverges**