3. [7 points] The function g defined by $g(x) = \ln(x^2+1)$ is differentiable for all x in $(-\infty, \infty)$. For all $x > 0$, the function $B(x) = \frac{1}{x}\int_0^x \ln(t^2+1)\,dt$ gives the average value of $g(x)$ over the interval $[0, x]$.
Note: Your answers may require one or more integral signs. However, neither the letter g nor the letter B should appear in your answers.

a. [4 points] Calculate $B'(x)$.

Solution: Using the product rule and the Construction Theorem (aka Second Fundamental Theorem of Calculus), we have

$$B'(x) = \left(\frac{d}{dx}\left(\frac{1}{x}\right)\right)\cdot\int_0^x \ln(t^2+1)\,dt + \frac{1}{x}\cdot\frac{d}{dx}\left(\int_0^x \ln(t^2+1)\,dt\right)$$
$$= \frac{-\int_0^x \ln(t^2+1)\,dt}{x^2} + \frac{\ln(x^2+1)}{x} = \frac{x\ln(x^2+1) - \int_0^x \ln(t^2+1)\,dt}{x^2}$$

Note that although it is possible to compute the integral $\int_0^x \ln(t^2+1)\,dt$, it is not necessary to do so here.

Answer: $B'(x) = \dfrac{x\ln(x^2+1) - \int_0^x \ln(t^2+1)\,dt}{x^2}$

b. [3 points] Write a formula for the average value of g' over the interval $[0, x]$.

Solution: (for $x \neq 0$)

$$\text{Average value of } g' \text{ over } [0,x] = \frac{1}{x-0}\int_0^x g'(t)\,dt = \frac{1}{x}\left(g(x) - g(0)\right)$$
$$= \frac{\ln(x^2+1) - \ln(0+1)}{x} = \frac{\ln(x^2+1)}{x}$$

Answer: Average value of g' over $[0, x]$ equals $\dfrac{\ln(x^2+1)}{x}$

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