3. [7 points] The function $g$ defined by $g(x) = \ln(x^2 + 1)$ is differentiable for all $x$ in $(-\infty, \infty)$. For all $x > 0$, the function $B(x) = \frac{1}{x} \int_0^x \ln(t^2 + 1) \, dt$ gives the average value of $g(x)$ over the interval $[0, x]$.

Note: Your answers may require one or more integral signs. However, neither the letter $g$ nor the letter $B$ should appear in your answers.

a. [4 points] Calculate $B'(x)$.

Solution: Using the product rule and the Construction Theorem (aka Second Fundamental Theorem of Calculus), we have

\[ B'(x) = \left( \frac{d}{dx} \left( \frac{1}{x} \right) \right) \cdot \int_0^x \ln(t^2 + 1) \, dt + \frac{1}{x} \cdot \frac{d}{dx} \left( \int_0^x \ln(t^2 + 1) \, dt \right) \]
\[ = - \int_0^x \frac{\ln(t^2 + 1)}{t^2} \, dt + \frac{\ln(x^2 + 1)}{x} = \frac{x \ln(x^2 + 1) - \int_0^x \ln(t^2 + 1) \, dt}{x^2} \]

Note that although it is possible to compute the integral $\int_0^x \ln(t^2 + 1) \, dt$, it is not necessary to do so here.

Answer: $B'(x) = \frac{x \ln(x^2 + 1) - \int_0^x \ln(t^2 + 1) \, dt}{x^2}$

b. [3 points] Write a formula for the average value of $g'$ over the interval $[0, x]$.

Solution: (for $x \neq 0$)

Average value of $g'$ over $[0, x] = \frac{1}{x - 0} \int_0^x g'(t) \, dt = \frac{1}{x} (g(x) - g(0))$
\[ = \frac{\ln(x^2 + 1) - \ln(0 + 1)}{x} = \frac{\ln(x^2 + 1)}{x} \]

Answer: Average value of $g'$ over $[0, x]$ equals $\frac{\ln(x^2 + 1)}{x}$.