7. [6 points] Split the function $\dfrac{5x^2-7x}{(x-1)^2(x+1)}$ into partial fractions with two or more terms. Do not integrate these terms. Be sure to show all work to obtain your partial fractions.

Solution: Let

$$\frac{5x^2-7x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Solution 1: To solve for B: multiply both sides by $(x-1)^2$, then put $x=1$:

$$\frac{5x^2-7x}{x+1} = A(x-1) + B + C\frac{x-1}{x+1},$$

$$\frac{-2}{2} = 0 + B + 0, B = -1.$$

To solve for C: multiply both sides by $x+1$, then put $x=-1$:

$$\frac{5x^2-7x}{(x-1)^2} = A\frac{x+1}{x-1} + B\frac{x+1}{(x-1)^2} + C,$$

$$\frac{12}{4} = 0 + 0 + C, C = 3.$$

To solve for A, clear the denominator, and compare the coefficients.

$$5x^2-7x = A(x-1)(x+1) - (x+1) + 3(x-1)^2 = (A+3)x^2 - 7x + (2-A).$$

Say we look at the coefficients of x^2:

$$5 = A+3, A = 2.$$

Therefore,

$$\frac{5x^2-7x}{(x-1)^2(x+1)} = \frac{2}{x-1} - \frac{1}{(x-1)^2} + \frac{3}{x+1}.$$

Solution 2: Clear the denominator.

$$5x^2-7x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 = (A+C)x^2 + (B-2C)x + (-A+B+C).$$

Compare the coefficients.

$$A+C=5, B-2C=-7, -A+B+C=0.$$

Solve the system of equations. An easy way is to write $A = 5-C$, $B = 2C-7$, and we then have $-(5-C)+(2C-7)+C=0$. We have that $A=2, B=-1, C=3$.

8. [13 points] Let $f(x)$ be a twice differentiable function with

- $f(0) = 1$.

- $f(\ln 2) = \frac{5}{4}$.
- $f'(0) = e$.
- $f'(\ln 2) = 2$.

a. [3 points] Compute the average value of $f'(x)$ on $[0, \ln 2]$.

Solution: The average value is

$$\frac{1}{\ln 2 - 0}\int_0^{\ln 2} f'(x)\, dx = \frac{1}{\ln 2}(f(\ln 2) - f(0)) = \frac{1}{\ln 2}(\frac{5}{4} - 1).$$

b. [5 points] Compute the exact value of $\int_0^{\ln 2} x f''(x)\, dx$.

Solution:

$$\begin{aligned}\int_0^{\ln 2} x f''(x)\, dx &= \left(x f'(x)\right)_0^{\ln 2} - \int_0^{\ln 2} f'(x)\, dx \\ &= (\ln 2 f'(\ln 2) - 0 f'(0)) - (f(\ln 2) - f(0)) \\ &= 2\ln 2 - \frac{5}{4} + 1.\end{aligned}$$

c. [5 points] Compute the exact value of $\int_0^{\ln 2} \frac{f'(x)}{\sqrt{9 - (f(x))^2}}\, dx$.

Solution: Let $w = f(x)$. The new upper and lower bounds are $f(0) = 1$ and $f(\ln 2) = \frac{5}{4}$ respectively.

$$\int_0^{\ln 2} \frac{f'(x)}{\sqrt{9 - (f(x))^2}}\, dx = \int_1^{\frac{5}{4}} \frac{1}{\sqrt{9 - w^2}}\, dw.$$

Let $w = 3\sin\theta$. The new θ-bounds are $\sin^{-1}\frac{1}{3}$ and $\sin^{-1}\frac{5}{12}$ respectively.

$$\begin{aligned}\int_1^{\frac{5}{4}} \frac{1}{\sqrt{9 - w}}\, dw &= \int_{\sin^{-1}\frac{1}{3}}^{\sin^{-1}\frac{5}{12}} \frac{3\cos\theta}{\sqrt{9 - 9\sin^2\theta}}\, d\theta \\ &= \int_{\sin^{-1}\frac{1}{3}}^{\sin^{-1}\frac{5}{12}} 1\, d\theta \\ &= \sin^{-1}\frac{5}{12} - \sin^{-1}\frac{1}{3}.\end{aligned}$$