9. [12 points] For each of the questions below, write out on your paper **all** the answers which are **always** true. For each answer you write, **give an explanation and/or a computation** that shows the statement is always true.

a. [6 points] Let $f(x)$ be a function defined for $0 \leq x \leq 1$ with $f(x) > 0$ and $f''(x) < 0$. Consider the Riemann sums for the integral $\int_0^1 f(x)\,dx$. Which of the following **must** be true?

LEFT(4) $\geq$ RIGHT(4)

MID(3) $\geq$ TRAP(2)

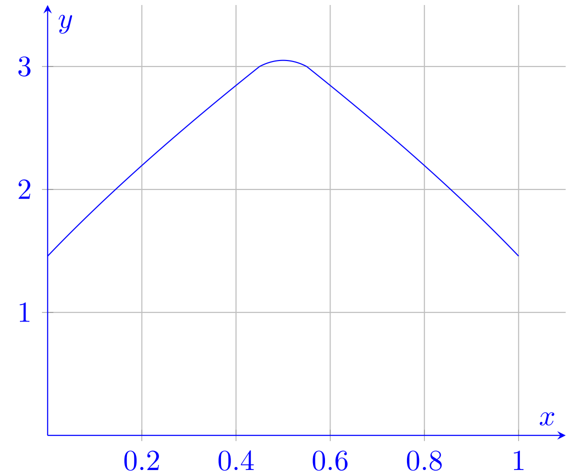
MID(3) is closer to the actual integral than MID(2) is

LEFT(3) $\leq$ TRAP(3)

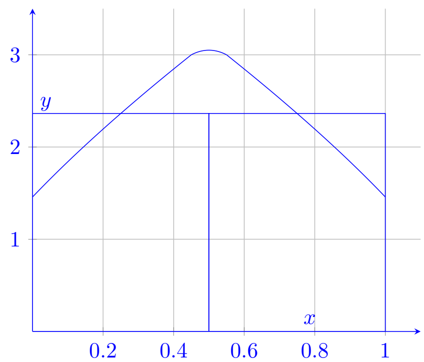
Solution: Only MID(3) $\geq$ TRAP(2) is always true. The function is concave down, so TRAP is an under-estimate and MID is an over-estimate. Hence MID(3) $\geq$ actual integral $\geq$ TRAP(2).

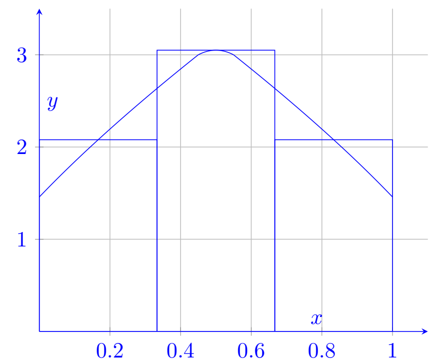
Extra explanation for why other choices are incorrect:
Choices involving LEFT or RIGHT: We do not know if $f(x)$ is increasing or decreasing, so we can't tell anything for LEFT or RIGHT.

Bottom left choice: In general, having more subdivisions does not guarantee that the estimates will always be better. What we can say is that the upper bound for the error will become smaller in the long run. However, in general it is always possible that a certain Riemann sum hits the exact value and some other Riemann sums with more subdivisions do not.
Even for concave up/down functions, it is possible that a certain MID with fewer subdivisions is better than another MID with more subdivisions. An actual graph of MID(2) better than MID(3) can be seen as below:



This graph is almost linear at $[0, 0.45]$ and $[0.55, 1]$ (you can see that it is still concave down if you zoom in a bit), and has a small hill around at $[0.45, 0.55]$. Since the graph is almost linear at $[0, 0.45]$, the first rectangle in MID(2) and the first rectangle in MID(3) is a good estimate of the respective area under the graph. Same applies to the last rectangles in MID(2) and MID(3). However, the middle rectangle of MID(3) is going to be overestimating by quite a bit. Therefore, MID(2) is a better estimate than MID(3).





b. [6 points] Let $g(x)$ be a differentiable function such that

- $0 < g(x) < 1$;
- $g'(x) \neq 0$.

Which of the following **must** be equal to $\dfrac{1}{\sin(g(x))}$?

$$\int_0^x \frac{g'(t)}{\sqrt{1-(g(t))^2}}\,dt \qquad \frac{d}{dx}\left(-\ln\left(\frac{1}{\sin(g(x))}+\frac{\cos(g(x))}{\sin(g(x))}\right)\right)$$

$$\frac{1}{g'(x)}\frac{d}{dx}\left(\int_1^{g(x)} \frac{1}{\sin t}\,dt\right) \qquad \int_0^x \frac{-g'(t)}{\sin(g(t))\tan(g(t))}\,dt + \frac{1}{\sin(g(0))}$$

Solution: $\dfrac{1}{g'(x)}\dfrac{d}{dx}\left(\displaystyle\int_1^{g(x)} \frac{1}{\sin t}\,dt\right)$ is correct, because

$$\frac{d}{dx}\left(\int_1^{g(x)} \frac{1}{\sin t}\,dt\right) = \frac{1}{\sin(g(x))}\cdot g'(x)$$

by chain rule.

$\displaystyle\int_0^x \frac{-g'(t)}{\sin(g(t))\tan(g(t))}\,dt + \frac{1}{\sin(g(0))}$ is correct. This can be seen by 2nd FTOC: this choice has the same values at $x = 0$ as $\dfrac{1}{\sin(g(x))}$ does, namely they are both $\dfrac{1}{\sin(g(0))}$. This choice also shares the same derivative as $\dfrac{1}{\sin(g(x))}$ does, namely

$$\frac{d}{dx}\frac{1}{\sin(g(x))} = \frac{-1}{\sin^2(g(x))}\cdot\cos(g(x))\cdot g'(x) = \frac{-g'(x)}{\sin(g(x))\tan(g(x))}.$$

Extra explanation for why other choices are incorrect:

$\displaystyle\int_0^x \frac{g'(t)}{\sqrt{1-(g(t))^2}}\,dt$ is not correct. It equals $\sin^{-1}(g(x))$, which is not the same as $\dfrac{1}{\sin(g(x))}$.

$\dfrac{d}{dx}\left(-\ln\left(\dfrac{1}{\sin(g(x))}+\dfrac{\cos(g(x))}{\sin(g(x))}\right)\right)$ is not correct, because this choice equals

$$\frac{1}{\sin(g(x))}\cdot g'(x).$$

A quick and easy way to rule this choice out (without actually calculating the derivative) is to notice that there will be a $g'(x)$ coming out from chain rule and no way to cancel it, but there is no $g'(x)$ in $\dfrac{1}{\sin(g(x))}$.