3. [12 points]

a. [6 points] Split the following function into partial fractions. Do not integrate the result. Be sure to show all your work.

$$\frac{10x+2}{(x-1)^2(x+2)}$$

Solution: Start by splitting:

$$\frac{10x+2}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

By giving terms a common denominator, we get:

$$10x+2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2. \qquad (1)$$

Method 1: (Comparing Coefficients) If we distribute terms, we get

$$0x^2 + 10x + 2 = (A+C)x^2 + (A+B-2C)x + (-2A+2B+C).$$

This gives the system of equations:

$$A+C=0 \qquad A+B-2C=10 \qquad -2A+2B+C=2$$

which we can solve for the values: $A=2$, $B=4$, $C=-2$.

Method 2: (Pluggin In Values) If we plug $x=1$ into (1) we get

$$10(1)+2 = A(1-1)(1+2) + B(1+2) + C(1-1)^2$$

This simplifies to give $12 = 3B$ and therefore $B = 4$.
If we plug in $x = -2$

$$10(-2)+2 = A(-2-1)(-2+2) + B(-2+2) + C(-2-1)^2$$

This gives $-18 = 9C$ and therefore $C = -2$.
Then, plugging in B, C and $x = -1$, we get:

$$\begin{aligned} 10(-1)+2 &= A(-1-1)(-1+2) + 4(-1+2) + (-2)(-1-1)^2 \\ &\Rightarrow -8 = -2A + 4 - 8 \\ &\Rightarrow -4 = -2A \Rightarrow A = 2. \end{aligned}$$

So we have $A=2$, $B=4$, $C=-2$.

b. [6 points] Given the partial fraction decomposition

$$\frac{-x-10}{(x-3)(x^2+4)} = \frac{-1}{(x-3)} + \frac{x+2}{x^2+4},$$

evaluate the following indefinite integral, show all your steps:

$$\int \frac{-x-10}{(x-3)(x^2+4)}dx$$

Solution: Start by substituting and splitting up the integral:

$$\int \frac{-x-10}{(x-3)(x^2+4)}dx = \int \frac{-1}{x-3}dx + \int \frac{x+2}{x^2+4}dx$$

Then we split up the second integral to get:

$$\int \frac{-x-10}{(x-3)(x^2+4)}dx = \int \frac{-1}{x-3}dx + \int \frac{x}{x^2+4}dx + \int \frac{2}{x^2+4}dx$$

For the first integral we have:

$$\int \frac{-1}{x-3}dx = -\ln|x-3| + C$$

For the second integral we use u-substitution with $u = x^2+4$ and $du = 2xdx$ to get

$$\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln(x^2+4) + C.$$

For the final integral, we first rewrite $\frac{2}{x^2+4} = \frac{2}{4((\frac{x}{2})^2+1)} = \left(\frac{1}{2}\right)\frac{1}{(\frac{x}{2})^2+1}$. Then we have:

$$\int \frac{2}{x^2+4}dx = \frac{1}{2}\int \frac{1}{(\frac{x}{2})^2+1}dx$$

Using substitution with $u = \frac{x}{2}$, so $du = \frac{1}{2}dx$, this becomes:

$$\int \frac{2}{x^2+4}dx = \frac{1}{2}\int \frac{1}{(\frac{x}{2})^2+1}dx = \int \frac{1}{u^2+1}du = \arctan(u) + C = \arctan\left(\frac{x}{2}\right) + C$$

Putting this all together, we get

$$\int \frac{-x-10}{(x-3)(x^2+4)}dx = -\ln|x-3| + \frac{1}{2}\ln(x^2+4) + \arctan\left(\frac{x}{2}\right) + C.$$