4. [11 points] Consider the shaded region bounded by $f(x) = 2 \cos^2 \left( \frac{\pi x}{4} \right)$ and $g(x) = \sqrt{4 - (x - 2)^2} + 2$ shown below.

![Graph](image)

**a.** [6 points] Write, but do not compute, an integral for the solid formed by rotating the region around the line $x = 5$.

**Solution:** Shell Method: The height of the shells are given by $h = g(x) - f(x)$, and the radius is the distance of the slice from the line $x = 5$. This is given by $r = 5 - x$. Since we are slicing along the $x$-axis for shell method, we integrate with respect to $x$ from 0 to 4. Using the shell method formula, we get:

$$V = \int_0^4 2\pi (5 - x) \left( \sqrt{4 - (x - 2)^2} + 2 - 2 \cos^2 \left( \frac{\pi x}{4} \right) \right) dx$$

Washer Method: First, we note that this is significantly more complicated than Shell, as both $f(x)$ and $g(x)$ must be inverted to obtain functions of $y$. Both are also non-one-to-one making inversion more tedious. Secondly, we must split the integral at $y = 2$. Therefore we have the following. On $[0, 2]$:

$r_{\text{in}}(y) = 5 - \frac{4}{\pi} \arccos \left(-\sqrt{\frac{y}{2}}\right)$  
$r_{\text{out}}(y) = 5 - \frac{4}{\pi} \arccos \left(\sqrt{\frac{y}{2}}\right)$

This means the first component of the volume integral is:

$$V_1 = \int_0^2 \pi \left( \left( 5 - \frac{4}{\pi} \arccos \left(\sqrt{\frac{y}{2}}\right) \right)^2 - \left(5 - \frac{4}{\pi} \arccos \left(-\sqrt{\frac{y}{2}}\right)\right)^2 \right) dy$$

On $[2, 4]$, we have:

$r_{\text{in}}(y) = 5 - \sqrt{4 - (y - 2)^2} + 2$  
$r_{\text{out}}(y) = 5 - \left(-\sqrt{4 - (y - 2)^2}\right) + 2$

giving the integral for this part as:

$$V_2 = \int_2^4 \pi \left( \left( 5 - \left(-\sqrt{4 - (y - 2)^2}\right) + 2 \right)^2 - \left(5 - \left(\sqrt{4 - (y - 2)^2} + 2\right)\right)^2 \right) dy$$

The final answer is then $V_1 + V_2$, whose full form has been omitted for brevity.
b. [5 points] Write, but do not compute, an expression involving one or more integrals for the perimeter of the region above. *Hint: The upper curve is a semicircle.*

**Solution:** We find the area of the two curves separately and add them.

Lower curve: The arclength formula is required. First find $f'(x)$ which is given by

$$2 \left( \frac{\pi}{4} \right) \left( \sin \left( \frac{\pi x}{4} \right) \right) \left( 2 \cos \left( \frac{\pi x}{4} \right) \right) = \pi \left( \sin \left( \frac{\pi x}{4} \right) \right) \left( \cos \left( \frac{\pi x}{4} \right) \right).$$

Then the arclength integral is given by:

$$L_1 = \int_0^4 \sqrt{1 + \left( \pi \sin \left( \frac{\pi x}{4} \right) \cos \left( \frac{\pi x}{4} \right) \right)^2} \, dx$$

Upper curve with the hint: The hint says the upper curve is a semicircle. The circle has radius 2, and so the circumference of the (full) circle is given by $4\pi$, with circumference of the semicircle given by $2\pi$.

Using the hint, the final answer is

$$L = \int_0^4 \sqrt{1 + \left( \pi \sin \left( \frac{\pi x}{4} \right) \cos \left( \frac{\pi x}{4} \right) \right)^2} \, dx + 2\pi$$

Upper curve without the hint: Use the arclength formula. First find $g'(x)$ as

$$g'(x) = \frac{(x - 2)}{\sqrt{4 - (x - 2)^2}}.$$

Then plug in the arclength formula to get

$$L_2 = \int_0^4 \sqrt{1 + \left( \frac{(x - 2)}{\sqrt{4 - (x - 2)^2}} \right)^2} \, dx$$

Without the hint, the final answer is:

$$L = \int_0^4 \sqrt{1 + \left( \pi \sin \left( \frac{\pi x}{4} \right) \cos \left( \frac{\pi x}{4} \right) \right)^2} \, dx + \int_0^4 \sqrt{1 + \left( \frac{(x - 2)}{\sqrt{4 - (x - 2)^2}} \right)^2} \, dx$$