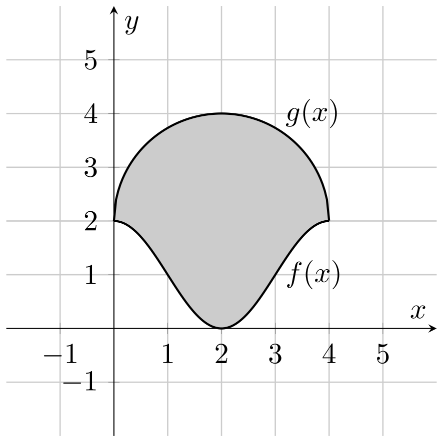
4. [11 points] Consider the shaded region bounded by $f(x) = 2\cos^2\left(\frac{\pi x}{4}\right)$ and $g(x) = \sqrt{4-(x-2)^2}+2$ shown below.



a. [6 points] Write, but do not compute, an integral for the solid formed by rotating the region around the line $x = 5$.

Solution: Shell Method: The height of the shells are given by $h = g(x) - f(x)$, and the radius is the distance of the slice from the line $x = 5$. This is given by $r = 5 - x$. Since we are slicing along the x-axis for shell method, we integrate with respect to x from 0 to 4. Using the shell method formula, we get:

$$V = \int_0^4 2\pi(5-x)\left(\sqrt{4-(x-2)^2}+2-2\cos^2\left(\frac{\pi x}{4}\right)\right)dx$$

Washer Method: First, we note that this is significantly more complicated than Shell, as both $f(x)$ and $g(x)$ must be inverted to obtain functions of y. Both are also non-one-to-one making inversion more tedious. Secondly, we must split the integral at $y = 2$. Therefore we have the following. On $[0, 2]$:

$$r_{\text{in}}(y) = 5 - \frac{4}{\pi}\arccos\left(-\sqrt{\frac{y}{2}}\right) \qquad r_{\text{out}}(y) = 5 - \frac{4}{\pi}\arccos\left(\sqrt{\frac{y}{2}}\right)$$

This means the first component of the volume integral is:

$$V_1 = \int_0^2 \pi\left(\left(5 - \frac{4}{\pi}\arccos\left(\sqrt{\frac{y}{2}}\right)\right)^2 - \left(5 - \frac{4}{\pi}\arccos\left(-\sqrt{\frac{y}{2}}\right)\right)^2\right)dy$$

On $[2, 4]$, we have:

$$r_{in}(y) = 5 - \sqrt{4-(y-2)^2}+2 \qquad r_{out}(y) = 5 - \left(-\sqrt{4-(y-2)^2}\right)+2$$

giving the integral for this part as:

$$V_2 = \int_2^4 \pi\left(\left(5-\left(-\sqrt{4-(y-2)^2}\right)+2\right)^2 - \left(5-\left(\sqrt{4-(y-2)^2}\right)+2\right)^2\right)dy$$

The final answer is then $V_1 + V_2$, whose full form has been omitted for brevity.

b. [5 points] Write, but do not compute, an expression involving one or more integrals for the perimeter of the region above. *Hint: The upper curve is a semicircle.*

Solution: We find the area of the two curves separately and add them.

Lower curve: The arclength formula is required. First find $f'(x)$ which is given by

$$2\left(\frac{\pi}{4}\right)\left(\sin\left(\frac{\pi x}{4}\right)\right)\left(2\cos\left(\frac{\pi x}{4}\right)\right) = \pi\left(\sin\left(\frac{\pi x}{4}\right)\right)\left(\cos\left(\frac{\pi x}{4}\right)\right).$$

Then the arclength integral is given by:

$$L_1 = \int_0^4 \sqrt{1 + \left(\pi \sin\left(\frac{\pi x}{4}\right)\cos\left(\frac{\pi x}{4}\right)\right)^2}\,dx$$

Upper curve with the hint: The hint says the upper curve is a semicircle. The circle has radius 2, and so the circumference of the (full) circle is given by 4π, with circumference of the semicircle given by 2π.

Using the hint, the final answer is

$$L = \int_0^4 \sqrt{1 + \left(\pi \sin\left(\frac{\pi x}{4}\right)\cos\left(\frac{\pi x}{4}\right)\right)^2}\,dx + 2\pi$$

Upper curve without the hint: Use the arclength formula. First find $g'(x)$ as

$$g'(x) = \frac{(x-2)}{\sqrt{4-(x-2)^2}}.$$

Then plug in the arclength formula to get

$$L_2 = \int_0^4 \sqrt{1 + \left(\frac{(x-2)}{\sqrt{4-(x-2)^2}}\right)^2}\,dx$$

Without the hint, the final answer is:

$$L = \int_0^4 \sqrt{1 + \left(\pi \sin\left(\frac{\pi x}{4}\right)\cos\left(\frac{\pi x}{4}\right)\right)^2}\,dx + \int_0^4 \sqrt{1 + \left(\frac{(x-2)}{\sqrt{4-(x-2)^2}}\right)^2}\,dx$$