8. [12 points] After constructing their boat, Brad and Shawna departed the island, hoping to return home. However, their victory at Troy angered Poseidon, the god of the sea, who created large waves to further complicate their journey. The waves were so high Brad and Shawna could not see land, making navigation difficult. Hermes, the messenger of the gods, was sympathetic. He stole the formula Poseidon used to create the waves and gave it to Brad and Shawna. The formula is given by:

$$H(t) = 12\int_0^{\sin^2(t)} e^{x^2}\,dx.$$

The function $H(t)$ gives the wave height in meters, t minutes after Hermes stole the formula.

a. [7 points] Brad and Shawna can only see land at the moment their ship is at the top of a wave. If they know exactly when that time is coming, they can be prepared to correct their course toward land. If Hermes brought them the formula 4 minutes after he stole it, when is the first time they can see land? (Brad and Shawna did not know when to look for land before obtaining the formula.)

Solution: We find the first maximum of $H(t)$ after $t = 4$. We need to find where $H'(t) = 0$. By applying chain rule and 2^{nd} Fundamental Theorem of calculus, we find

$$H'(t) = 24\sin(t)\cos(t)e^{\sin^4(t)}$$

Since e^x is always positive, $e^{\sin^4(t)}$ can never be zero, so the critical points are where $\sin(t) = 0$ and $\cos(t) = 0$. These are $n\pi$ for $\sin(t)$ and $n\pi + \frac{\pi}{2}$ for $\cos(t)$. Now, we need to determine which are maximum and which are minimum. Since $\sin(n\pi) = 0$, if we plug these into $H(t)$, we see that $H(n\pi) = \int_0^0 e^{x^2}\,dx = 0$. If we check $n\pi + \frac{\pi}{2}$, $\sin(n\pi + \frac{\pi}{2}) = \pm 1$ so $\sin^2(n\pi + \frac{\pi}{2}) = 1$ and so $H(n\pi + \frac{\pi}{2}) = \int_0^1 e^{x^2}\,dx$, which is positive since e^{x^2} is always positive. Therefore, these are the maximum. If we start counting, we see that $\frac{\pi}{2} \approx 1.5$ and $\frac{3\pi}{2} \approx 4.5$ so the answer is $\frac{3\pi}{2}$.

b. [5 points] The same poet who recorded the tale of the Trojan war would like to record parts of Brad and Shawna's odyssey. Seeking a more appealing version of the expression above, he asks you for a different formula. Write a function equivalent to $H(t)$ with only t in the upper bound of the integral.

Solution: Using second fundamental theorem, we integrate $H'(t)$, which we solved for above. This gives

$$H(t) = \int_0^t 24\sin(x)\cos(x)e^{\sin^4(x)}\,dx$$