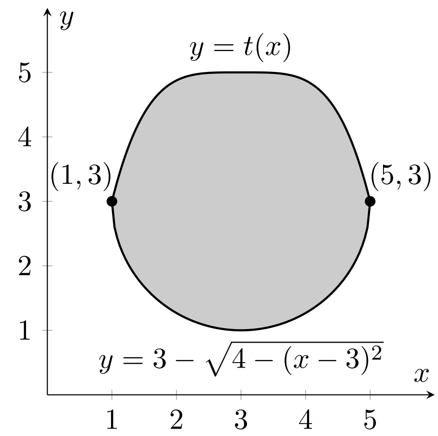
**6**. [16 points] Ariana, a baker at the vegan bakery VCorp, is designing a new doughnut. The cross section of the doughnut is shown below, where the units of both $x$ and $y$ are cm.




The top of the cross section is given by the function $y = t(x)$ and the bottom is given by the semicircle $y = 3 - \sqrt{4 - (x-3)^2}$. Ariana is experimenting with two ideas for the doughnut.

**a**. [6 points] Her first idea is to rotate the cross section around the $y$-axis. Write an integral that gives the volume of the resulting doughnut. Do not evaluate your integral. Your answer may involve the function $t$, but it should not involve $t^{-1}$ (the inverse of $t$).

*Solution:* We use the shell method here. For each vertical slice of thickness $\Delta x$ (cm), we have $\Delta V = 2\pi x(t(x) - (3 - \sqrt{4-(x-3)^2}))\Delta x$ (cm$^3$). The volume of the resulting doughnut (in cm$^3$) is given by integrating the above expression from $x = 1$ to $x = 5$.

**Answer:** $\int_1^5 2\pi x(t(x) - (3 - \sqrt{4-(x-3)^2}))\, dx$ **cm**$^3$

**b**. [6 points] Her second idea is to rotate the cross section around the $x$-axis. Write an integral that gives the volume of the resulting doughnut. Do not evaluate your integral. Your answer may involve the function $t$, but it should not involve $t^{-1}$ (the inverse of $t$).

*Solution:* We use the washer method here. For each vertical slice of thickness $\Delta x$ (cm), we have $\Delta V = \pi((t(x))^2 - (3 - \sqrt{4-(x-3)^2})^2)\Delta x$ (cm$^3$). The volume of the resulting doughnut (in cm$^3$) is given by integrating the above expression from $x = 1$ to $x = 5$.

**Answer:** $\int_1^5 \pi((t(x))^2 - (3 - \sqrt{4-(x-3)^2})^2)\, dx$ **cm**$^3$

**c**. [4 points] As is tradition at VCorp, they are planning to wrap a ribbon around the cross section of the doughnut. Write an expression involving one or more integrals that gives the total perimeter of the cross section. Do not evaluate the integrals in your expression.

*Solution:* We find the length of the top and bottom halves of the cross section separately, and add them. For the top curve, $y' = t'(x)$. And, for the bottom curve, $y' = \dfrac{x-3}{\sqrt{4-(x-3)^2}}$. Therefore, using the arclength formula $\displaystyle\int_{x=a}^{x=b} \sqrt{1+(y')^2}\,dx$, with $a = 1$ and $b = 5$ for each of these pieces and adding them, we get the total perimeter (in cm) of the cross section,

$$\int_1^5 \sqrt{1+(t'(x))^2}\,dx + \int_1^5 \sqrt{1+\left(\frac{x-3}{\sqrt{4-(x-3)^2}}\right)^2}\,dx.$$

The length of the bottom piece can also be directly calculated to be $2\pi$, by noting that it is a semicircle of radius 2.

**Answer:** $\displaystyle\int_1^5 \sqrt{1+(t'(x))^2}\,dx + 2\pi$ **cm**