7. [10 points] Jamal is refurbishing a watch and he is placing small white sapphires around the watch face. The existing space on the watch face can accept regularly shaped sapphires of volume (in cubic mm):

\[ V(t) = \int_{-\frac{1}{2}}^{1-t} \sqrt{1-x^2} \, dx \]

for any number \( t \) satisfying \( 0 \leq t \leq 1 \).

a. [4 points] Compute \( V'(t) \). Your expression should not involve any integrals.

\[ \text{Solution:} \quad V'(t) = \sqrt{1-(1-t)^2} \cdot (-1) - \sqrt{1-\left(-\frac{t}{2}\right)^2} \cdot \left(-\frac{1}{2}\right). \]

Answer: \( V'(t) = \frac{1}{2} \sqrt{1-(1-t)^2} - \sqrt{1-\left(-\frac{t}{2}\right)^2} \).
b. [6 points] Jamal would like to know the volume of the smallest sapphire he can use on the watch face. Given that $V(t)$ has its only critical point at $t \approx \frac{2}{15}$, find the $t$-value(s) in the interval $0 \leq t \leq 1$ where the minimum of $V(t)$ occurs. Justify your answer using the fact that the graph of $y = \sqrt{1-x^2}$ is the top half of the unit circle centered at the origin.

Solution: First, we recall that the global minimum of $V(t)$ on $0 \leq t \leq 1$ can only occur at any critical point(s) in the interval or at the endpoint(s) of the interval.

As we are provided that $t \approx \frac{2}{15}$ is the only critical point, we will first attempt to classify it as a local min/max. For this, we invoke the first derivative test with the value of $V'$ on either side of $t \approx \frac{2}{15}$, say at $t = 0$ and $t = 1$ for computational convenience. Using our result in part a., we calculate $V'(0) = \frac{1}{2} > 0$, and $V'(1) = -1 + \frac{1}{2} \sqrt{\frac{3}{4}} < 0$. Therefore, by the first derivative test, $V(t)$ has a local maximum at $t \approx \frac{2}{15}$. Therefore, it cannot be a global minimum.

The only other candidates, for the global minimum, that remain are the endpoints of the interval, namely $t = 0$ and $t = 1$. We note that $V(0) = \int_0^1 \sqrt{1-x^2} \, dx$ is the area of quarter of the unit circle, whereas $V(1) = \int_{-\frac{1}{2}}^0 \sqrt{1-x^2} \, dx$ describes an area that is less than that of the quarter unit circle. Therefore, $V(1) < V(0)$, and hence, $t = 1$ must be the global minimum of $V(t)$.

Alternatively, we can directly compare the values $V(0)$, $V\left(\frac{2}{15}\right)$, and $V(1)$.

- $V(0) = \int_0^1 \sqrt{1-x^2} \, dx$ is the area of quarter of the unit circle.
- $V(1) = \int_{-\frac{1}{2}}^0 \sqrt{1-x^2} \, dx$ describes an area that is less than that of the quarter unit circle.
- $V\left(\frac{2}{15}\right) = \int_{-\frac{13}{30}}^{\frac{13}{30}} \sqrt{1-x^2} \, dx$ includes the interval $[0, \frac{1}{2}]$ which, due to symmetry, implies that $V\left(\frac{2}{15}\right)$ describes an area that is larger than $V(1)$.

As $V(1)$ is smaller than both $V(0)$ and $V\left(\frac{2}{15}\right)$, by the Extreme Value Theorem, $t = 1$ must be the global minimum of $V(t)$.

Answer: The minimum of $V(t)$ occurs at $t = 1$. 