7. [10 points] Jamal is refurbishing a watch and he is placing small white sapphires around the watch face. The existing space on the watch face can accept regularly shaped sapphires of volume (in cubic mm):

$$V(t) = \int_{-\frac{t}{2}}^{1-t} \sqrt{1-x^2}\, dx$$

for any number t satisfying $0 \le t \le 1$.

a. [4 points] Compute $V'(t)$. Your expression should not involve any integrals.

Solution: By the chain rule and the second FTC, we have

$$V'(t) = \sqrt{1-(1-t)^2} \cdot (-1) - \sqrt{1-\left(-\frac{t}{2}\right)^2} \cdot \left(\frac{-1}{2}\right).$$

Answer: $V'(t) =$ $-\sqrt{1-(1-t)^2} + \frac{1}{2}\sqrt{1-\left(-\frac{t}{2}\right)^2}$.

b. [6 points] Jamal would like to know the volume of the smallest sapphire he can use on the watch face. Given that $V(t)$ has its only critical point at $t \approx \frac{2}{15}$, find the t-value(s) in the interval $0 \leq t \leq 1$ where the minimum of $V(t)$ occurs. Justify your answer using the fact that the graph of $y = \sqrt{1-x^2}$ is the top half of the unit circle centered at the origin.

Solution: First, we recall that the *global* minimum of $V(t)$ on $0 \leq t \leq 1$ can only occur at any critical point(s) in the interval or at the endpoint(s) of the interval.

As we are provided that $t \approx \frac{2}{15}$ is the only critical point, we will first attempt to classify it as a local min/max. For this, we invoke the first derivative test with the value of V' on either side of $t \approx \frac{2}{15}$, say at $t = 0$ and $t = 1$ for computational convenience. Using our result in part **a.**, we calculate $V'(0) = \frac{1}{2} > 0$, and $V'(1) = -1 + \frac{1}{2}\sqrt{\frac{3}{4}} < 0$. Therefore, by the first derivative test, $V(t)$ has a local maximum at $t \approx \frac{2}{15}$. Therefore, it cannot be a global minimum.

The only other candidates, for the global minimum, that remain are the endpoints of the interval, namely $t = 0$ and $t = 1$. We note that $V(0) = \int_0^1 \sqrt{1-x^2}\,dx$ is the area of quarter of the unit circle, whereas $V(1) = \int_{-\frac{1}{2}}^{0} \sqrt{1-x^2}\,dx$ describes an area that is less than that of the quarter unit circle. Therefore, $V(1) < V(0)$, and hence, $t = 1$ must be the global minimum of $V(t)$.

Alternatively, we can directly compare the values $V(0)$, $V(\frac{2}{15})$, and $V(1)$.

- $V(0) = \int_0^1 \sqrt{1-x^2}\,dx$ is the area of quarter of the unit circle.
- $V(1) = \int_{-\frac{1}{2}}^{0} \sqrt{1-x^2}\,dx$ describes an area that is less than that of the quarter unit circle.
- $V(\frac{2}{15}) = \int_{-\frac{2}{30}}^{\frac{13}{15}} \sqrt{1-x^2}\,dx$ includes the interval $[0, \frac{1}{2}]$ which, due to symmetry, implies that $V(\frac{2}{15})$ describes an area that is larger than $V(1)$.

As $V(1)$ is smaller than both $V(0)$ and $V(\frac{2}{15})$, by the Extreme Value Theorem, $t = 1$ must be the global minimum of $V(t)$.

Answer: The minimum of $V(t)$ occurs at $t =$ **1**.