3. [18 points] In the video game *The Legend of Helga*, the heroes Helga and Lank must save the world from the evil wizard Kanon. In the final battle, Helga and Kanon duel each other, while Lank creates magical spells to assist Helga.

a. [4 points] Using a magical spell costs Magic Points (MP), of which Lank has a limited amount. Let $L(t)$ be the amount of MP that Lank has remaining t minutes after the battle starts. The function $L(t)$ is differentiable, and its **derivative** $L'(t)$ is given by

$$L'(t) = (1 - t^3)^{1/3}.$$

Suppose that, after 5 minutes of battle, Lank has 20 MP remaining.
Write a formula involving an integral for $L(t)$. Your answer should not involve the letter L.

Solution: By the Construction Theorem, $L(t)$ can be written in the form

$$L(t) = C + \int_a^t (1 - x^3)^{1/3}\, dx$$

for some constants a and C. Since Lank has 20 MP remaining after 5 minutes of battle, this means that $L(5) = 20$, so we can set $a = 5$ and $C = 20$ to obtain our formula.

Answer: $L(t) =$ $20 + \int_5^t (1 - x^3)^{1/3}\, dx$

b. [5 points] While casting magical spells, Lank dodges Kanon's attacks by running along a path modeled by the curve

$$H(x) = \frac{1}{6}(4x - 1)^{3/2}$$

from $x = 1$ to $x = 16$, where distances are measured in meters.
Compute the **exact value** of the **arc length** of this curve. You need to evaluate all integrals completely, and your answer should not involve the letter H, but you do not need to simplify your final answer. Show all your work.

Solution: Recall that the arc length of $H(x)$ from $x = 1$ to $x = 16$ is given by the formula

$$\int_1^{16} \sqrt{1 + H'(x)^2}\, dx.$$

By the power rule and the chain rule, the formula for $H'(x)$ is given by

$$H'(x) = (4x - 1)^{1/2},$$

so the arc length integral is

$$\int_1^{16} \sqrt{1 + H'(x)^2}\, dx = \int_1^{16} \sqrt{1 + 4x - 1}\, dx = 2\int_1^{16} \sqrt{x}\, dx.$$

The integral can be evaluated using a standard antiderivative:

$$2\int_1^{16} \sqrt{x}\, dx = \frac{4}{3}x^{3/2}\Big|_1^{16} = \frac{4}{3}(16^{3/2} - 1^{3/2}) = 84.$$

This is our answer.

Answer: 84 meters

This problem continues on the next page.

3. (continued)

As Helga attacks Kanon throughout the battle, Kanon's Hit Points (HP) decrease. The amount of HP that Kanon has t minutes after the battle starts is given by the differentiable function

$$K(t) = \int_{t-4}^{\sqrt{t+16}} \left(12 + \cos(w^2)\right) dw.$$

c. [5 points] Compute the **exact value** of $K'(8)$. You do not need to simplify your final answer.

Solution: First we rewrite the given function $K(t)$ as

$$K(t) = \int_0^{\sqrt{t+16}} \left(12 + \cos(w^2)\right) dw - \int_0^{t-4} \left(12 + \cos(w^2)\right) dw.$$

By the Second Fundamental Theorem of Calculus and the chain rule, we obtain

$$K'(t) = \frac{1}{2\sqrt{t+16}} \left(12 + \cos\left((\sqrt{t+16}\,)^2\right)\right) - \left(12 + \cos\left((t-4)^2\right)\right).$$

Substituting $t = 8$, we obtain our answer.

Answer: $K'(8) = \dfrac{1}{2\sqrt{24}} \left(12 + \cos(24)\right) - \left(12 + \cos(16)\right)$

d. [4 points] Kanon is defeated upon reaching 0 HP, at which point the battle is over. Find the **one** value of t at which Kanon is defeated. Show all of your work.

Hint: If you find multiple such values of t, check which ones are actually solutions.

Solution: We have $K(t) = 0$ when $t - 4 = \sqrt{t+16}$, so we solve for t:

$$\begin{aligned} t - 4 &= \sqrt{t+16} \\ (t-4)^2 &= t + 16 \\ t^2 - 8t + 16 &= t + 16 \\ t^2 - 9t &= 0 \\ (t-9)t &= 0 \end{aligned}$$

This gives the "solutions" $t = 0$ and $t = 9$, but observe that $t = 0$ is an extraneous solution of the original equation because $0 - 4 \neq \sqrt{0+16}$. On the other hand, $t = 9$ is a solution since $9 - 4 = \sqrt{9+16}$.

Answer: $t = 9$