4. [9 points] The city of Rainneapolis has a strange weather pattern. It is always sunny, except for February 2nd, when it rains substantially all day. This year, Amin prepared for the stormy day by building a machine which continuously removes rainwater from his backyard.

- Let $T(h)$ be the **total amount** of rainwater in Amin's backyard, in cubic feet, h hours after 12:00am on February 2nd.
- Let $A(h)$ be the **rate** at which the rain **adds** water to Amin's backyard, in cubic feet per hour, h hours after 12:00am on February 2nd.
- Let $M(h)$ be the **rate** at which Amin's **machine removes** rainwater from his backyard, in cubic feet per hour, h hours after 12:00am on February 2nd.

The functions $T(h)$, $A(h)$, and $M(h)$ are all differentiable. Assume that there is **no rainwater** in Amin's backyard **before** it starts raining at 12:00am on February 2nd.

a. [3 points] Which of the following gives a correct interpretation of $\int_4^{10} M(h)\,dh = 8000$? Circle **all** correct answers.

(i) The total amount of rainwater in Amin's backyard decreases by 8000 cubic feet from 4:00am to 10:00am on February 2nd.

(ii) Between 4:00am and 10:00am on February 2nd, Amin's machine removes a total of 8000 cubic feet of rainwater from his backyard.

(iii) The rate at which Amin's machine removes rainwater from his backyard between 4:00am and 10:00am on February 2nd is 8000 cubic feet per hour.

(iv) At 10:00am on February 2nd, Amin's machine removes rainwater from his backyard at a rate of 8000 cubic feet per hour faster than at 4:00am.

(v) NONE OF THESE

Solution: The function $M(h)$ is the **rate** at which Amin's machine **removes** rainwater from his backyard, so its definite integral $\int_4^{10} M(h)\,dh$ gives the **amount** of rainwater that is **removed** from his backyard between 4:00am and 10:00am. Hence $\int_4^{10} M(h)\,dh = 8000$ means that the total amount of rainwater that is removed from his backyard between 4:00am and 10:00am is 8000 cubic feet, so (ii) is the answer. To explain why the other choices are incorrect:

First, (i) is incorrect because it is a statement about the **total amount** of rainwater in Amin's backyard, not the amount that is **removed.** The given equation only involves $M(h)$, which models water removal. Choice (i) describes the equation $T(10) - T(4) = -8000$, or $\int_4^{10} T'(h)\,dh = -8000$.

Next, (iii) is incorrect because it is a statement about the **rate** at which water is removed, not a statement about the **amount** of water removed. Also, (iii) suggests that this removal rate is a constant 8000 cubic feet per hour between 4:00am and 10:00am, which is not supported by the problem. Choice (iii) describes the statement "$M(h) = 8000$ for $4 \leq h \leq 10$."

Finally, (iv) is incorrect for the same reason: it describes the **rate**, not the **amount.** Choice (iv) describes the equation $M(10) - M(4) = 8000$, or $\int_4^{10} M'(h)\,dh = 8000$.

b. [3 points] Which of the following expressions gives the **total amount** of rainwater, in cubic feet, in Amin's backyard at 7:00am? Circle **all** correct answers.

[(i)] $\int_0^7 T'(h)\,dh$ (ii) $\int_0^7 T(h)\,dh$ (iii) $\int_0^7 \big(A(h)+M(h)\big)\,dh$

[(iv)] $\int_0^7 A(h)\,dh - \int_0^7 M(h)\,dh$ (v) NONE OF THESE

Solution: By definition of $T(h)$, the total amount of rainwater in Amin's backyard at 7:00am equals $T(7)$. So we are looking for quantities that equal $T(7)$. Note that $T'(h) = A(h) - M(h)$ by definition of rates, so by the First Fundamental Theorem of Calculus,

$$\int_0^7 A(h)\,dh - \int_0^7 M(h)\,dh = \int_0^7 (A(h) - M(h))\,dh = \int_0^7 T'(h)\,dh = T(7) - T(0) = T(7),$$

where $T(0) = 0$ because there is no rainwater in Amin's backyard before it starts raining at 12:00am on February 2nd. The above equation shows that (i) and (iv) are correct.

Now, (ii) is incorrect because $\int_0^7 T(h)\,dh$ is the **integral** of the total amount of rainwater, not the amount itself. Also, (iii) is incorrect because $\int_0^7 \big(A(h)+M(h)\big)\,dh$ is the amount of rainwater that falls in Amin's backyard **plus** the amount that is removed by 7:00am, which does not give the total amount of rainwater at 7:00am.

c. [3 points] Which of the following expressions gives the **average amount** of rainwater, in cubic feet, in Amin's backyard between 6:00am and 9:00am? Circle **all** correct answers.

(i) $\frac{1}{9-6}\int_6^9 T'(h)\,dh$ [(ii)] $\frac{1}{9-6}\int_6^9 T(h)\,dh$ (iii) $\frac{T(9)-T(6)}{9-6}$

[(iv)] $\frac{1}{3}\int_0^9 T(h)\,dh + \frac{1}{3}\int_6^0 T(h)\,dh$ (v) NONE OF THESE

Solution: First note that, by properties of integrals,

$$\frac{1}{3}\int_0^9 T(h)\,dh + \frac{1}{3}\int_6^0 T(h)\,dh = \frac{1}{3}\int_0^9 T(h)\,dh - \frac{1}{3}\int_0^6 T(h)\,dh = \frac{1}{3}\int_6^9 T(h)\,dh,$$

and by the First Fundamental Theorem of Calculus,

$$\frac{1}{9-6}\int_6^9 T'(h)\,dh = \frac{T(9)-T(6)}{9-6}.$$

The first equation shows that (ii) and (iv) are equivalent, and the second equation shows that (i) and (iii) are equivalent.

Now, (ii) is correct, because this is the formula for the **average value** of $T(h)$, the **amount** of rainwater, between 6:00am and 9:00am. So (iv) is also correct. On the other hand, (iii) is incorrect, as this is the **average rate of change** of the amount of rainwater from 6:00am to 9:00am, which is different from the **average value** in that time frame. So (i) is also incorrect.