5. [14 points]

a. [6 points] Split the following expression into partial fractions with two or more terms. **Do not integrate these terms.** Please clearly show all of your work.

$$\frac{5x-4}{(x-2)^2(x+1)}$$

Solution: The partial fraction decomposition of the given function has the form

$$\frac{5x-4}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}.$$

By multiplying to obtain a common denominator, we have

$$5x-4 = A(x-2)(x+1) + B(x+1) + C(x-2)^2. \qquad (*)$$

Below are two possible ways to complete the problem:

Solution 1 (Comparing coefficients)**:** By multiplying out the terms on the right-hand side of the equation (*) and grouping terms with the same powers of x, we obtain

$$0x^2 + 5x - 4 = (A+C)x^2 + (-A+B-4C)x + (-2A+B+4C).$$

This gives the system of equations

$$A + C = 0, \qquad -A + B - 4C = 5, \qquad -2A + B + 4C = -4.$$

The first equation implies $C = -A$, so the second and third equations become

$$3A + B = 5, \qquad -6A + B = -4.$$

Taking this first equation and subtracting the second equation from it, we obtain $9A = 9$. Thus $A = 1$, so $3A + B = 5$ implies $B = 2$, and $C = -A$ implies $C = -1$. This gives us the answer.

Solution 2 (Plugging in values)**:** By setting $x = 2$ in the equation (*), we obtain

$$5(2) - 4 = A(2-2)(2+1) + B(2+1) + C(2-2)^2.$$

This simplifies to $6 = 0 + 3B + 0$, thus $B = 2$. Now setting $x = -1$ in (*), we obtain

$$5(-1) - 4 = A(-1-2)(-1+1) + B(-1+1) + C(-1-2)^2.$$

This simplifies to $-9 = 0 + 0 + 9C$, thus $C = -1$. Finally, setting $x = 0$ in (*) and using the facts that $B = 2$ and $C = -1$, we obtain

$$5(0) - 4 = A(0-2)(0+1) + 2(0+1) + (-1)(0-2)^2.$$

This simplifies to $-4 = -2A + 2 - 4$, thus $A = 1$. This gives us the answer.

Answer: $\dfrac{1}{x-2} + \dfrac{2}{(x-2)^2} + \dfrac{-1}{x+1}$

b. [8 points] Use the partial fraction decomposition

$$\frac{4x-2}{(3-x)(x^2+1)} = \frac{1}{3-x} + \frac{x-1}{x^2+1}$$

to evaluate the following **indefinite** integral. Please clearly show all of your work.

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx.$$

Solution: Using the partial fraction decomposition:

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx = \int \frac{1}{3-x}\,dx + \int \frac{x-1}{x^2+1}\,dx.$$

Now we use linearity:

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx = \int \frac{1}{3-x}\,dx + \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx.$$

The first integral is a standard antiderivative (or, substitute $u = 3-x$):

$$\int \frac{1}{3-x}\,dx = -\ln|3-x| + C.$$

The second integral can be evaluated using the substitution $u = x^2+1$, so that $du = 2x\,dx$, and thus

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C.$$

The third integral is a standard antiderivative:

$$-\int \frac{1}{x^2+1}\,dx = -\arctan(x) + C.$$

Putting this all together, our final answer is

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx = -\ln|3-x| + \frac{1}{2}\ln|x^2+1| - \arctan(x) + C.$$

Answer: $-\ln|3-x| + \frac{1}{2}\ln|x^2+1| - \arctan(x) + C$