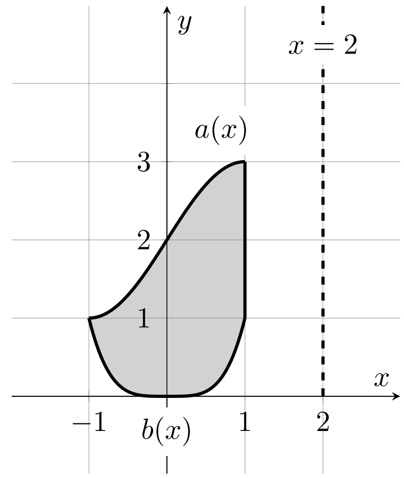
6. [11 points] Louise, a world-famous abstract artist and cheese enthusiast, is experimenting with new designs for cheese sculptures. She has two ideas for a cheese sculpture and would like to know the volume of each one so that she knows how much cheese to buy.

a. [6 points] Louise's first idea involves the shaded region to the right, which is bounded by the line $x = 1$ and the curves

$$a(x) = 2 + \sin\left(\tfrac{\pi}{2}x\right) \qquad \text{and} \qquad b(x) = x^4$$

on the interval $[-1, 1]$.



Write an integral that represents the volume of the solid formed by rotating this region around the line $x = 2$. **Do not evaluate your integral.** Your answer should not involve the letters a or b.

Solution: There are two ways to solve this problem:

Solution 1 (Shell method)**:** The shell method states that the volume of a cylindrical shell of thickness Δx at a point $-1 \le x \le 1$ is of the form $2\pi r(x)h(x)\,\Delta x$ for some functions $r(x)$ and $h(x)$, representing the radius and height of our cylindrical shell, respectively.

For each $-1 \le x \le 1$, the distance from x to the line $x = 2$ is given by $2 - x$, so $r(x) = 2 - x$ is the radius of our shell. The height of our shell is $h(x) = a(x) - b(x)$, that is, $h(x) = 2 + \sin(\frac{\pi}{2}x) - x^4$. This gives us our answer:

$$\int_{-1}^{1} 2\pi(2 - x)\left(2 + \sin\left(\frac{\pi}{2}x\right) - x^4\right) dx$$

Solution 2 (Washer method)**:** Using the washer method is considerably more difficult: we must invert both $a(x)$ and $b(x)$, and we must split the integral at $y = 1$. Inverting these functions:

$$y = 2 + \sin\left(\frac{\pi}{2}x\right) \qquad \Rightarrow \qquad x = \frac{2}{\pi}\arcsin(y - 2),$$

$$y = x^4 \qquad \Rightarrow \qquad x = \pm y^{1/4}.$$

The washer method states that the volume of a washer of thickness Δy at a point $0 \le y \le 3$ is of the form $\pi(R(y)^2 - r(y)^2)\,\Delta y$ for some functions $R(y)$ and $r(y)$, representing the outer radius and inner radius of our washer, respectively.

For $0 \le y \le 1$, the outer radius is $R(y) = 2 + y^{1/4}$, and the inner radius is $r(y) = 2 - y^{1/4}$.
For $1 \le y \le 3$, the outer radius is $R(y) = 2 - \frac{2}{\pi}\arcsin(y - 2)$, and the inner radius is $r(y) = 1$.

Putting this all together, this gives us our answer:

$$\int_0^1 \pi\left((2 + y^{1/4})^2 - (2 - y^{1/4})^2\right) dy + \int_1^3 \pi\left(\left(2 - \frac{2}{\pi}\arcsin(y - 2)\right)^2 - 1^2\right) dy$$

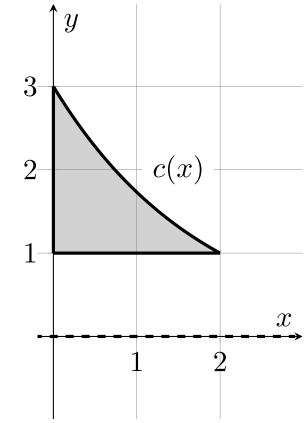
Answer: $\displaystyle\int_{-1}^{1} 2\pi(2 - x)\left(2 + \sin\left(\frac{\pi}{2}x\right) - x^4\right) dx$

b. [5 points] Louise's second idea involves the shaded region to the right, bounded by the curve

$$c(x) = \left(\sqrt{3}\right)^{2-x},$$

the y-axis, and the line $y = 1$ on the interval $[0, 2]$.

Write an integral that represents the volume of the solid formed by rotating this region around the x-axis. **Do not evaluate your integral.** Your answer should not involve the letter c.



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For each $0 \leq x \leq 2$, the outer radius $R(x)$ is given by the distance from the x-axis to $y = c(x)$, which is $R(x) = (\sqrt{3})^{2-x}$. The inner radius $r(x)$ is given by the distance from the x-axis to $y = 1$, which is $r(x) = 1$. This gives us our answer:

$$\int_0^2 \pi \left(\left((\sqrt{3})^{2-x} \right)^2 - 1^2 \right) dx$$

Solution 2 (Shell method)**:** The shell method requires an additional step, but here it is not too complicated. First we invert the function $c(x)$:

$$y = (\sqrt{3})^{2-x} \qquad \Rightarrow \qquad x = 2 - \frac{\ln(y)}{\ln(\sqrt{3})}.$$

The shell method states that the volume of a cylindrical shell of thickness Δy at a point $1 \leq y \leq 3$ is of the form $2\pi r(y)h(y)\,\Delta y$ for some functions $r(y)$ and $h(y)$, representing the radius and height of our cylindrical shell, respectively.

For each $1 \leq y \leq 3$, the radius of our shell is $r(y) = y$, and the height of our shell is $h(y) = 2 - \frac{\ln(y)}{\ln(\sqrt{3})}$. This gives us our answer:

$$\int_1^3 2\pi y \left(2 - \frac{\ln(y)}{\ln(\sqrt{3})} \right) dy$$

Answer: $\int_0^2 \pi \left(\left((\sqrt{3})^{2-x} \right)^2 - 1^2 \right) dx$