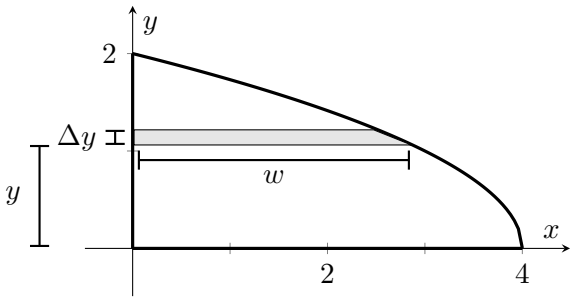
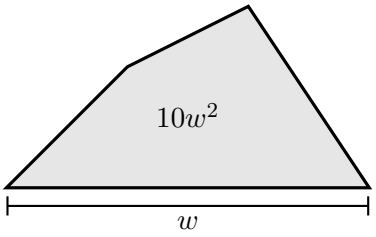
7. [11 points] In an accidental discovery, scientists created the Ultra Bouncy Toy (UBT), which bounces unpredictably due to its unusual shape and irregular density.
The base of the UBT is the region bounded by $y = \sqrt{4 - x}$, the x-axis, and the y-axis, shown below to the left. All distances are measured in centimeters (cm). A sample slice of the base of width w and thickness Δy is shown in the graph below to the left. Cross-sections of the UBT perpendicular to the ***y*-axis** have the shape shown below to the right. The area of such a cross-section is $10w^2$.





a. [3 points] Write a formula in terms of y for the width w of a slice that is y centimeters above the x-axis. **Include units.**

Solution: We are given $y = \sqrt{4 - w}$. Solving for w, we have $w = 4 - y^2$.

Answer: $w =$ $4 - y^2$ **Units:** **cm**

b. [3 points] Write an expression that approximates the **volume of a slice** of the UBT that is y centimeters above the x-axis and has thickness Δy centimeters. Your answer should not involve the letter w. **Include units.**

Solution: The approximate volume of a slice is given by (volume) = (area of cross-section) Δy. Using part (a), the area of a cross-section is $10w^2 = 10(4 - y^2)^2$. Thus the volume of a slice is about $10(4 - y^2)^2\,\Delta y$ cm^3.

Answer: $10(4 - y^2)^2\,\Delta y$ **Units:** **cm^3**

The density of the UBT is given by the function $\delta(y)$, measured in grams per cubic centimeter (g/cm^3), where y is the distance from the x-axis in centimeters.

c. [2 points] Write an expression that approximates the **mass of a slice** of the UBT that is y centimeters above the x-axis and has thickness Δy centimeters. Your answer may include δ, but it should not involve the letter w. **Include units.**

Solution: The approximate mass of a slice is given by (mass) = (volume)(density). At a distance y, the density in an entire slice is approximately $\delta(y)$ g/cm^3. By part (b), the volume of a slice is $10(4 - y^2)^2\,\Delta y$ cm^3. Thus the mass of a slice is about $10(4 - y^2)^2\delta(y)\,\Delta y$ g.

Answer: $10(4 - y^2)^2\delta(y)\,\Delta y$ **Units:** **g**

d. [3 points] Write an expression involving an integral that represents the **total mass** of the UBT. Your answer may include δ. **Include units.**

Solution: By part (c), the mass of a slice is approximately $10(4-y^2)^2\delta(y)\,\Delta y$. The total mass is obtained by taking the integral of the mass of all slices as $\Delta y \to 0$ for $0 \le y \le 2$. This gives our answer in the form of an integral.

Answer: $\displaystyle\int_0^2 10(4 - y^2)^2\delta(y)\,dy$ **Units:** **g**