8. [6 points] Each part below describes a twice differentiable function and one or more approximations of its integral. For each of the following statements, determine if the statement is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true, and circle the appropriate answer.
No justification is required.

a. [1 point] If $A'(x) > 0$ for all x, then $\text{LEFT}(4) \le \int_{-1}^{1} A(x)\,dx$.

Circle one: $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

Solution: Since $A'(x) > 0$, then $A(x)$ is always increasing on $[-1, 1]$, so LEFT(4) is an underestimate of $\int_{-1}^{1} A(x)\,dx$.

b. [1 point] If $B'(x) > 0$ for all x, then $\text{TRAP}(4) \le \int_{-1}^{1} B(x)\,dx$.

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

Solution: Since $B'(x) > 0$, then $B(x)$ is always increasing on $[-1, 1]$. But we are not given whether $B(x)$ is concave up or concave down on $[-1, 1]$, so TRAP(4) could be an underestimate (for example, if $B(x) = -e^{-x}$) or an overestimate (for example, if $B(x) = e^{x}$) of $\int_{-1}^{1} B(x)\,dx$.

c. [1 point] If $C''(x) > 0$ for all x, then $\text{TRAP}(4) \le \int_{-1}^{1} C(x)\,dx$.

Circle one: **ALWAYS** **SOMETIMES** $\boxed{\textbf{NEVER}}$

Solution: Since $C''(x) > 0$, then $C(x)$ is always concave up on $[-1, 1]$, so TRAP(4) is an overestimate of $\int_{-1}^{1} C(x)\,dx$, not an underestimate. Moreover, since $C''(x) > 0$, then $C(x)$ has no inflection points, so we cannot even have $\text{TRAP}(4) = \int_{-1}^{1} C(x)\,dx$.

d. [1 point] If $D(x)$ is odd and MID(4) approximates $\int_{-1}^{1} D(x)\,dx$, then $\text{MID}(4) = 0$.

Circle one: $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

Solution: By direct computation, $\text{MID}(4) = D(-\frac{3}{4}) + D(-\frac{1}{4}) + D(\frac{1}{4}) + D(\frac{3}{4})$. But $D(x)$ is odd, so $D(-\frac{3}{4}) = -D(\frac{3}{4})$ and $D(-\frac{1}{4}) = -D(\frac{1}{4})$, therefore $\text{MID}(4) = 0$.

e. [1 point] If $E'(x) > 0$ and $E''(x) < 0$ for all x, then $\displaystyle\int_{-1}^{1} E(x)\,dx \le \mathrm{MID}(2) \le \mathrm{RIGHT}(2)$.

Circle one: $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

Solution: Since $E'(x) > 0$ and $E''(x) < 0$, then $E(x)$ is always increasing and concave down on $[-1, 1]$, so both RIGHT(2) and MID(2) are overestimates of $\displaystyle\int_{-1}^{1} E(x)\,dx$. Moreover, $E(x)$ always increasing implies that MID(2) is a better approximation of the integral than RIGHT(2). More specifically, we note that

$$\mathrm{MID}(2) = E(-\tfrac{1}{2}) + E(\tfrac{1}{2}) \qquad \text{and} \qquad \mathrm{RIGHT}(2) = E(0) + E(1),$$

and since $E(x)$ is always increasing, then $E(-\frac{1}{2}) \le E(0)$ and $E(\frac{1}{2}) \le E(1)$, so therefore

$$\mathrm{MID}(2) = E(-\tfrac{1}{2}) + E(\tfrac{1}{2}) \le E(0) + E(1) = \mathrm{RIGHT}(2).$$

f. [1 point] If $F(x)$ is not constant, then RIGHT(3) approximates the integral $\displaystyle\int_{-1}^{1} F(x)\,dx$ more accurately than RIGHT(2).

Circle one: **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

Solution: If $F(x) = x$, then $\mathrm{RIGHT}(2) = 1$ and $\mathrm{RIGHT}(3) = \dfrac{2}{3}$, while $\displaystyle\int_{-1}^{1} F(x)\,dx = 0$, so RIGHT(3) is a more accurate estimate than RIGHT(2). On the other hand, consider the function

$$F(x) = \begin{cases} 1 & x \le 0, \\ 1 - 6(3x)^5 + 15(3x)^4 - 10(3x)^3 & 0 < x < \frac{1}{3}, \\ 0 & x \ge \frac{1}{3}. \end{cases}$$

(Draw a picture!) By direct computation, $\mathrm{RIGHT}(2) = 1$ and $\mathrm{RIGHT}(3) = \dfrac{2}{3}$, but $\displaystyle\int_{-1}^{1} F(x)\,dx > 1$, so RIGHT(2) is a more accurate estimate than RIGHT(3) in this case. (Note that this shows the statement can be false even if $F(x)$ never increases.)