2. [15 points] Let $f(x)$ be an **odd**, **twice-differentiable** function defined for all real numbers. Some values of $f(x)$ and $f'(x)$ are given in the table below:

x	1	2	3	4	5
$f(x)$	3	8	5	0	7
$f'(x)$	2	0	-1	-2	6

Compute the exact value of the following quantities. If there is not enough information provided to answer the question, write "NEI" and clearly indicate why. Show all of your work.

a. [5 points] $\displaystyle\int_{e^{-2}}^{e} \frac{f'(1+\ln x)}{x}\,\mathrm{d}x$.

Solution: We use the substitution,

$$u = 1 + \ln x \qquad \mathrm{d}u = \frac{1}{x}\mathrm{d}x.$$

Note that

$$x = e^{-2} \implies u = 1 + \ln(e^{-2}) = 1 - 2 = -1$$
$$x = e \implies u = 1 + \ln e = 1 + 1 = 2.$$

Then

$$\int_{e^{-2}}^{e} \frac{f'(1+\ln x)}{x}\,\mathrm{d}x = \int_{-1}^{2} f'(u)\,\mathrm{d}u = f(2) - f(-1)$$

where we have used First Fundamental Theorem of Calculus. Since $f(x)$ is odd, $f(-1) = -f(1) = -3$. Therefore, we obtain

$$\int_{e^{-2}}^{e} \frac{f'(1+\ln x)}{x}\,\mathrm{d}x = f(2) - f(-1) = 8 - (-3) = 11$$

Answer: 11

b. [5 points] $\int_1^4 f(x)e^{f(x)}f'(x)\,\mathrm{d}x$.

Solution:
Method 1. We set

$$u = f(x) \qquad \mathrm{d}u = f'(x)\mathrm{d}x$$
$$\mathrm{d}v = e^{f(x)}f'(x)\mathrm{d}x \qquad v = e^{f(x)}$$

Now, integrating by parts,

$$\begin{aligned}\int_1^4 f(x)e^{f(x)}f'(x)\,\mathrm{d}x &= f(x)e^{f(x)}\Big|_1^4 - \int_1^4 f'(x)e^{f(x)}\,\mathrm{d}x\\ &= f(x)e^{f(x)}\Big|_1^4 - e^{f(x)}\Big|_1^4\\ &= \left(f(x)e^{f(x)} - e^{f(x)}\right)\Big|_1^4\\ &= \left(f(4)e^{f(4)} - e^{f(4)}\right) - \left(f(1)e^{f(1)} - e^{f(1)}\right)\\ &= (0e^0 - e^0) - (3e^3 - e^3)\\ &= -1 - 2e^3\end{aligned}$$

Method 2. We use the substitution

$$y = f(x) \qquad \mathrm{d}y = f'(x)\mathrm{d}x$$

Then

$$\int_1^4 f(x)e^{f(x)}f'(x)\,\mathrm{d}x = \int_3^0 ye^y\,\mathrm{d}y.$$

We set

$$u = y \qquad \mathrm{d}u = \mathrm{d}y$$
$$\mathrm{d}v = e^y\mathrm{d}y \qquad v = e^y$$

Now, integrating by parts,

$$\begin{aligned}\int_3^0 ye^y\,\mathrm{d}y &= ye^y\Big|_3^0 - \int_3^0 e^y\,\mathrm{d}y\\ &= ye^y\Big|_3^0 - e^y\Big|_3^0\\ &= (ye^y - e^y)\Big|_3^0\\ &= (0e^0 - e^0) - (3e^3 - e^3)\\ &= -1 - 2e^3\end{aligned}$$

Answer: $-1 - 2e^3$

c. [5 points] $\int_1^9 f''(\sqrt{x})\,\mathrm{d}x$.

Solution: We use the substitution

$$y = \sqrt{x} \qquad \mathrm{d}y = \frac{1}{2\sqrt{x}}\mathrm{d}x = \frac{1}{2y}\mathrm{d}x$$

Then

$$\int_1^9 f''(\sqrt{x})\,\mathrm{d}x = 2\int_1^3 y f''(y)\,\mathrm{d}y$$

Now, integrating by parts,

$$\begin{aligned}\int_1^3 y f''(y)\,\mathrm{d}y &= y f'(y)\Big|_1^3 - \int_1^3 f'(y)\,\mathrm{d}y \\ &= \big(y f'(y) - f(y)\big)\Big|_1^3 \\ &= \big(3f'(3) - f(3)\big) - \big(1f'(1) - f(1)\big) \\ &= (3(-1) - 5) - (2 - 3) \\ &= -7\end{aligned}$$

Therefore,

$$\int_1^9 f''(\sqrt{x})\,\mathrm{d}x = 2\int_1^3 y f''(y)\,\mathrm{d}y = -14.$$

Answer: -14