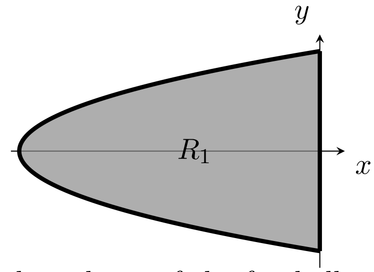
6. [10 points] Justin, a member of the 2023 undefeated Michigan football team, is graduating this semester. He wants to celebrate by building a replica of the Rose Bowl trophy that is twice the size of the original. The trophy consists of two components: a silver football and a plinth.

a. [5 points]

Let R_1 be the region bounded by the y-axis and the curve $x = \left(\frac{y}{2}\right)^2 - 36$, as shown to the right. To form the silver football, Justin rotates this region around the y-axis.



Find an expression involving one or more integrals for the volume of the football using **horizontal** slices of the region R_1. **Do not** evaluate any integrals in your expression.

Solution: We use horizontal slices to solve this problem. Consider a thin horizontal slice of region R_1, located at y, with a small thickness Δy. When this slice is rotated about the y-axis, it forms a disk. The radius of the disk, denoted by R, is given by

$$R = 36 - \left(\frac{y}{2}\right)^2$$

The approximate volume of this disk is

$$\pi R^2 \,\Delta y = \pi \left(36 - \left(\frac{y}{2}\right)^2\right)^2 \Delta y.$$

Note that the domain of the slices are given by the y-intercepts of the curve $\left(\frac{y}{2}\right)^2 - 36$, which yields $y = -12$ and $y = 12$. By integrating from $y = -12$ to $y = 12$, we obtain the total volume of the football:

$$V = \int_{-12}^{12} \pi \left(36 - \left(\frac{y}{2}\right)^2\right)^2 \mathrm{d}y.$$

Answer: $\int_{-12}^{12} \pi \left(36 - \left(\frac{y}{2}\right)^2\right)^2 dy$

b. [5 points]

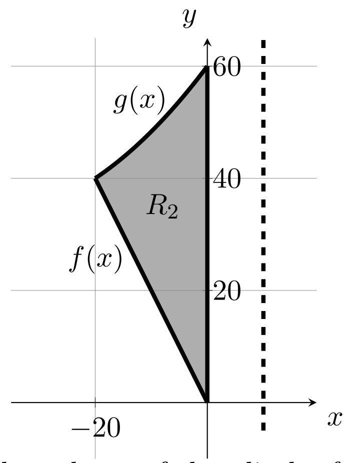
Consider the shaded region R_2 depicted to the right. The region is bounded by $y = f(x)$, $y = g(x)$, and the y-axis, where

$$g(x) = \frac{1}{60}\left((x+40)^2 + 2000\right),$$

and

$$f(x) = -2x.$$

The plinth for the trophy is formed by rotating the region R_2 about the vertical line $x = 10$.



Find an expression involving one or more integrals for the volume of the plinth of the trophy using **vertical** slices of the region R_2. Your answer should not involve the letters f or g. **Do not** evaluate any integrals in your expression.

Solution: We use vertical slices to solve this problem. Consider a thin vertical slice of region R_2, located x units to the left of the line $x = 10$, with a small thickness Δx. When this slice is rotated about the $x = 10$, it forms a shell. The height and radius of the shell, denoted by h and r, are given by

$$h = g(x) - f(x) = \frac{1}{60}\left((x+40)^2 + 2000\right) + 2x \quad \text{and} \quad r = 10 - x$$

Hence, the approximate volume of this shell is

$$2\pi r h \Delta x = 2\pi \left(\frac{1}{60}\left((x+40)^2 + 2000\right) + 2x\right)(10 - x)\,\Delta x$$

To determine the domain of x, we solve the equation $f(x) = g(x)$, which gives two answers $x = -20$ or $x = 0$. Hence, we obtain the total volume of the plinth:

$$V = \int_{-20}^{0} 2\pi \left(\frac{1}{60}\left((x+40)^2 + 2000\right) + 2x\right)(10 - x)\,dx$$

Answer: $\displaystyle\int_{-20}^{0} 2\pi \left(\frac{1}{60}\left((x+40)^2 + 2000\right) + 2x\right)(10 - x)\,dx$