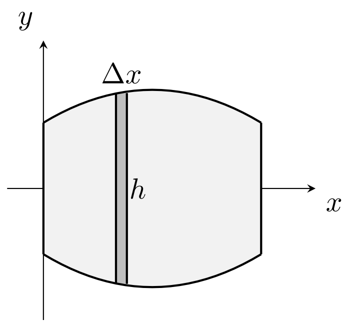
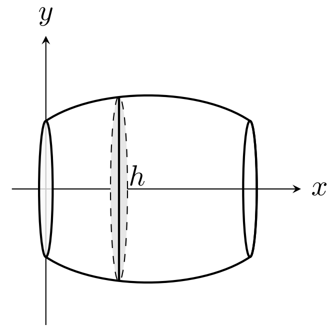
7. [10 points] Gragas, a brewmaster, gifts his friend Braum a barrel **full** of whiskey. Braum wants to determine the total mass of the alcohol stored inside.

The side view of the barrel, shown below and to the left, is bounded by the vertical lines $x = 0$ and $x = 60$, and the curves $y = 30 - \frac{1}{90}(x-30)^2$ and $y = -30 + \frac{1}{90}(x-30)^2$. All lengths are measured in centimeters.





a. [2 points] Consider a thin vertical strip of the barrel located x cm from the y-axis, as shown in the diagram above and to the left. This strip has a vertical length h. Find a formula for h in terms of x.

Solution: The length h is given by the difference of the top and bottom curves:

$$h = 30 - \frac{1}{90}(x-30)^2 - \left(-30 + \frac{1}{90}(x-30)^2\right) = 60 - \frac{2}{90}(x-30)^2$$

Answer: $h =$ $60 - \frac{2}{90}(x-30)^2$

b. [4 points] Cross-sections of the barrel perpendicular to the x-axis are circles. Consider a thin cylindrical slice located x cm from the y-axis, with thickness Δx. Find an expression for the volume of that slice. Your answer should **not** involve any integrals, and should **not** include the letter h. Include units.

Solution: Note that the radius of the cylindrical slice, denoted by r, is given by

$$r = \frac{h}{2} = 30 - \frac{1}{90}(x-30)^2$$

Therefore, the approximate volume of the slice is given by

$$\pi r^2 \,\Delta x = \pi\left(30 - \frac{1}{90}(x-30)^2\right)^2 \Delta x$$

Answer: $\pi\left(60 - \frac{2}{90}(x-30)^2\right)^2 \Delta x$ **Units:** cm^3

c. [4 points] Braum determines that the density of the liquid varies along the length of the barrel according to the function $\rho(x) = 0.01x + 1$, measured in g/cm^3.

Write an expression involving one or more integrals that represents the total mass of the whiskey inside the barrel. **Do not** evaluate any integrals in your expression. Your answer should **not** involve the letters h or ρ. Include units.

Solution: To find the mass of the cylindrical slice from part b. we multiply its volume by the density function. We see its mass is

$$(0.01x+1)\pi\left(60-\frac{2}{90}(x-30)^2\right)^2\Delta x$$

Integrating from $x=0$ to $x=60$, we get the total mass

$$m=\int_0^{60}(0.01x+1)\pi\left(30-\frac{1}{90}(x-30)^2\right)^2\,dx$$

Answer: $\int_0^{60}(0.01x+1)\pi\left(30-\frac{1}{90}(x-30)^2\right)^2\,dx$ **Units:** g