8. [11 points] Consider the following function $y(t)$ defined for $t \geq 0$ by:

$$y(t) = 3 + e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx$$

a. [2 points] Let $y_0 = y(\pi)$. Determine the value of y_0.

Solution: Evaluate $y(t)$ at $t = \pi$, we get:

$$y(\pi) = 3 + e^{-\pi} \int_0^{\sin(\pi)} \frac{1}{1+x^4} dx = 3 + e^{-\pi} \int_0^{0} \frac{1}{1+x^4} dx = 3 + 0.$$

Answer: $y_0 =$ 3

b. [4 points] Compute the derivative $y'(t)$. Your answer may involve an integral.

Solution: Apply chain rule and product rule:

$$\begin{aligned} y'(t) =& \frac{d}{dt}(e^{-t}) \int_0^{\sin(t)} \frac{1}{1+x^4} dx + e^{-t} \frac{d}{dt} \left(\int_0^{\sin(t)} \frac{1}{1+x^4} dx \right) \\ =& -e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx + e^{-t} \frac{1}{1+\sin(t)^4} \cos(t) \end{aligned}$$

Answer: $y'(t) =$ $-e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx + e^{-t} \frac{\cos(t)}{1+\sin(t)^4}$

c. [2 points] The function $y(t)$ is a *solution* to a first-order differential equation of the form:

$$e^t y'(t) + e^t y(t) = f(t) \quad \text{for } t \geq 0 \tag{$\star$}$$

Substitute your expressions for $y(t)$ and $y'(t)$ into the left-hand side of $(\star)$. Simplify the result to find an explicit formula for $f(t)$. Your final expression for $f(t)$ should **not** involve any integrals.

Solution: Use the expression for $y(t)$ and $y'(t)$, we get

$$\begin{aligned} e^t y'(t) + e^t y(t) =& e^t \left(3 + e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx \right) \\ &+ e^t \left(-e^{-t} \int_0^{\sin(t)} \frac{1}{1+x^4} dx + e^{-t} \frac{\cos(t)}{1+\sin(t)^4} \right) \\ =& 3e^t + \frac{\cos(t)}{1+\sin(t)^4} \end{aligned}$$

Answer: $f(t) =$ $3e^t + \frac{\cos(t)}{1+\sin(t)^4}$

d. [3 points] It can be shown that we may rewrite the solution $y(t)$ in the following form:

$$y(t) = e^{-t}\left(\int_a^t f(x)\,dx + C\right)$$

where a and C are constants, and $f(x)$ is the expression you found in part **c**. Find values of the constants a and C so that the solution satisfies $y(\pi) = y_0$, where y_0 is the value you found in part **a.**

Solution: Using the fact $y(\pi) = 3$, we see

$$e^{-\pi}\left(\int_a^\pi f(x)\,dx + C\right) = 3$$

Pick $a = \pi$ to eliminate the integral, we get $e^{-\pi}C = 3$, hence $C = 3e^{-\pi}$.

Answer: $a =$ π and $C =$ $3e^{-\pi}$