7. (14 points) Suppose the amount of time x that riders wait for the bus to arrive at a certain bus stop is given by the probability density function

$$f(x) = \frac{1}{10}e^{-\frac{1}{10}x}$$

where x is measured in minutes.

(a) What percentage of the time will a rider wait less than 5 minutes for the bus to arrive? (Show your work.)

Knowing the formula to use is the most important part here and should be given most of the credit. It is ok to use their calculators to calculate the integral, and integration errors should not have a large number of points deducted.

The percentage of time the rider will wait less then 5 minutes is given by the probability that the waiting time is less then 5 minutes. This is given by

$$\begin{aligned}
\textit{Waiting time} < 5\,\textit{minutes} &= \int_0^5 f(x)dx \\
&= \frac{1}{10}\int_0^5 e^{-\frac{1}{10}x}dx \\
&= \int_0^{\frac{1}{2}} e^{-u}du \\
&= 1 - e^{-\frac{1}{2}} \\
&= 0.39
\end{aligned}$$

So 39% *of the time a rider will wait less then 5 minutes for the bus.*

(b) What is the mean waiting time until the next bus arrives? (Show your work.)

The mean waiting time is given by

$$\begin{aligned}
\textit{mean waiting time} &= \int_0^\infty xf(x)dx \\
&= \frac{1}{10}\int_0^\infty xe^{-\frac{1}{10}x}dx \\
&= \lim_{b\to\infty} -xe^{-\frac{1}{10}x}\Big|_0^b + \frac{1}{10}\int_0^\infty 10e^{-\frac{1}{10}x}dx \qquad \textit{(integration by parts)} \\
&= 10\int_0^\infty e^{-u}du \\
&= 10 \quad \textit{minutes.}
\end{aligned}$$

(c) What is the median waiting time until the next bus arrives? (Show your work)

The median waiting time T is given by

$$\begin{aligned}
0.5 &= \int_0^T f(x)dx \\
&= \int_0^T \frac{1}{10}e^{-\frac{1}{10}x}dx \\
&= 1 - e^{-\frac{1}{10}T}
\end{aligned}$$

Solving for T we get that $T = -10\ln\frac{1}{2} \simeq 6.93$ minutes.