3. (10 points)

(a) Find the radius of convergence R of the following power series. *Show your work.*

$$\sum_{n=1}^{\infty} \frac{(n + n^3\, 2^n)}{n^2\, 3^n}\,(x-1)^n\,.$$

The general coefficient of the given power series is $a_n = \dfrac{n + n^3\, 2^n}{n^2\, 3^n}(x-1)^n$. *We need to find the limit as* n *goes to infinity of the ratio* $\left|a_{n+1}/a_n\right|$. *For this, observe that for large values of* n, *the term* a_n *"behaves" like* $\dfrac{n^3\, 2^n}{n^2\, 3^n}\,(x-1)^n = n\,(2/3)^n (x-1)^n$. *Thus we have:*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \;=\; \lim_{n\to\infty}\frac{(n+1)(2/3)^{n+1}\,|x-1|^{n+1}}{n(2/3)^n\,|x-1|^n} \;=\; \frac{2}{3}\lim_{n\to\infty}\frac{n+1}{n}\,|x-1| \;=\; \frac{2}{3}\,|x-1|\,.$$

Therefore, the radius of convergence of the power series is $\mathbf{\dfrac{3}{2}}$.

(b) What is the interval of convergence of the series?

The power series is given around the point $x = 1$, *and we have found its radius of convergence to be* $3/2$. *Accordingly, the series converges for values of* x *within the point* $1 - 3/2 = -1/2$, *and the point* $1 + 3/2 = 5/2$.

The interval of convergence is therefore: $-\mathbf{\dfrac{1}{2}} < \mathbf{x} < \mathbf{\dfrac{5}{2}}$.