3. (10 points)

(a) Find the radius of convergence $R$ of the following power series. Show your work.

$$
\sum_{n=1}^{\infty} \frac{(n + n^3 2^n)}{n^2 3^n} (x - 1)^n.
$$

The general coefficient of the given power series is $a_n = \frac{n + n^3 2^n}{n^2 3^n} (x - 1)^n$. We need to find the limit as $n$ goes to infinity of the ratio $|a_{n+1}/a_n|$. For this, observe that for large values of $n$, the term $a_n$ “behaves” like $\frac{n^3 2^n}{n^2 3^n} (x - 1)^n = n (2/3)^n (x - 1)^n$. Thus we have:

$$
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{n + 1}{n} \frac{(2/3)^{n+1}}{(2/3)^n} \frac{|x - 1|^{n+1}}{|x - 1|^n} \right| = \frac{2}{3} \lim_{n \to \infty} \frac{n + 1}{n} |x - 1| = \frac{2}{3} |x - 1|.
$$

Therefore, the radius of convergence of the power series is $\frac{3}{2}$.

(b) What is the interval of convergence of the series?

The power series is given around the point $x = 1$, and we have found its radius of convergence to be $3/2$. Accordingly, the series converges for values of $x$ within the point $1 - 3/2 = -1/2$, and the point $1 + 3/2 = 5/2$.

The interval of convergence is therefore: $-\frac{1}{2} < x < \frac{5}{2}$. 