5. (10 points)

(a) Find the second order Taylor polynomial of the function $f(x) = \sqrt{4+x}$ for x near 0. *You must show the calculations that lead to your answer.*

We will need two derivatives of $f(x)$. *It's not hard to compute* $f'(x) = \frac{1}{2}\big(x+4\big)^{-1/2}$, *and* $f''(x) = -\frac{1}{4}\big(x+4\big)^{-3/2}$. *Hence* $f'(0) = 1/4$ *and* $f''(0) = -1/32$. *Since in addition* $f(0) = 2$, *we obtain the second-order Taylor polynomial for* $f(x)$ *near* $x = 0$*:*

$$P_2(x) \;=\; f(0) \,+\, f'(0)\,x \,+\, \frac{f''(0)}{2}\,x^2 \;=\; \mathbf{2} + \frac{\mathbf{x}}{\mathbf{4}} - \frac{\mathbf{x^2}}{\mathbf{64}}\,.$$

(b) What is the Taylor series about $x = 0$ of the function $\sin x$? *No explanation required.*

This series is well-known. Near $x = 0$, *we have*

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,.$$

(c) Using your answers to parts **(a)** and **(b)** and *without computing any derivatives*, find the second order Taylor polynomial that approximates $g(x) = \sqrt{4+\sin(2x)}$ for x near 0. *Show your work.*

Of course, there's no need to differentiate $g(x)$ *to derive the polynomial. Indeed, observe that* $g(x) = f\big(sin(2x)\big)$ *; so we can make use of our answers to parts* **(a)** *and* **(b)**.

From part **(b)**, *we see that* $\sin(2x) \approx 2x$ *for* x *near 0. Thus, for small values of* x, *the function* $\sin(2x)$ *has values near zero. Thus, from part* **(a)**, *we deduce:*

$$g(x) \;\approx\; P_2\big(\sin(2x)\big) \;=\; 2 + \frac{\sin(2x)}{4} - \frac{\sin^2(2x)}{64}\,.$$

Next, from part **(b)**, *we get:*

$$\sin(2x) \;\approx\; 2x - \frac{(2x)^3}{3!}\,.$$

Substituting this in the approximation we found for $g(x)$ *gives us:*

$$g(x) \;\approx\; 2 + \frac{2x - 8x^3/3!}{4} - \frac{\big(2x - 8x^3/3!\big)^2}{64}\,.$$

To finish the work, there remains to truncate the latter to second order. Hence we obtain:

$$g(x) \;\approx\; 2 + \frac{2x}{4} - \frac{(2x)^2}{64} \;=\; \mathbf{2} + \frac{\mathbf{x}}{\mathbf{2}} - \frac{\mathbf{x^2}}{\mathbf{16}}\,,$$

which is the desired second-order Taylor polynomial near $x = 0$ *for the function* $g(x)$.