8. (13 points) We shall investigate a well-known physical phenomenon, called the "Doppler Effect". When an electromagnetic signal (e.g. a ray of light) with frequency F_e is emitted from a source moving away with velocity $v > 0$ with respect to a receiver at rest, then the received frequency F_r is different from F_e. The relationship linking the emitted frequency F_e and the received frequency F_r is the Doppler Law:

$$F_r = \sqrt{\frac{1-v/c}{1+v/c}}\, F_e \ , \qquad \text{where } c \text{ is a constant, the speed of light.}$$

For this problem, you might find useful to know that the third order Taylor polynomial for the function $\sqrt{\frac{1+x}{1-x}}$ near $x = 0$ is $1 + x + \frac{x^2}{2} + \frac{x^3}{2}$.

(a) On Earth, nearly all objects travel with velocities v much smaller than the speed of light c, i.e. the ratio v/c is very small. Use this fact to obtain the Doppler Law for slow-moving emitters:

$$F_r \approx \left(1 - \frac{v}{c}\right) F_e \, .$$

If we substitute $-v/c$, which we are told is very small, for x in the given Taylor polynomial, we obtain:

$$\begin{aligned} F_r &= \sqrt{\frac{1-v/c}{1+v/c}}\, F_e = \left(1 - \frac{v}{c} + \frac{(-v/c)^2}{2} + \frac{(-v/c)^3}{2}\right) F_e + \cdots \\ &= \left(1 - \frac{v}{c}\right) F_e + \frac{v^2}{2c^2} F_e - \frac{v^3}{2c^3} F_e + \cdots \, . \end{aligned}$$

Truncating the latter gives the desired approximation for slow-moving emitters.

(b) The relationship in part **(a)** is *not* exact, and an error is made when it is used to approximate the Doppler Law. Find an expression for the "error", in terms of v, c and F_e. Is the approximation accurate within 1% of F_e if the velocity is at most 10% of the speed of light c? *Explain.*

The error made when approximating the Doppler Law by the relationship given in part **(a)** *is the sum of infinitely many powers of v/c. We found above the first two of these terms are* $\frac{v^2}{2c^2} F_e$ *and* $\frac{-v^3}{2c^3} F_e$.

Accordingly, we may use $\frac{v^2}{2c^2} F_e$ *as a good approximation for the error.*

If the velocity is at most 10% of the speed of light, then $v/c \le 0.1$. Hence we deduce

$$\textit{Approximate Error} \le \frac{1}{2}(0.1)^2 F_e = 0.005\, F_e \, .$$

Thus we find the "error" to be less than half of 1% of F_e, which is indeed within the suggested bound.