3. (14 points) Please note that the two parts of this problem involve different power series.

(a) (8 pts.) Use the ratio test to find the *radius of convergence*, R, for the series

$$\sum_{n=0}^{\infty} \frac{3^n}{n!} x^n.$$

Show step-by-step work.

We use the ratio test. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3^{n+1}}{(n+1)!}x^{n+1}}{\frac{3^n}{n!}x^n}\right| = |x| \; \frac{3^{n+1}}{(n+1)!} \; \frac{n!}{3^n} = |x|\frac{3}{n+1}.$$

Since

$$\lim_{n\to\infty} |x|\frac{3}{n+1} = |x| \lim_{n\to\infty} \frac{3}{n+1} = 0,$$

the radius of convergence is $R = \infty$. That is, this series converges for all x in $(-\infty, \infty)$.

(b) (6 pts.) The power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{2^n n^3},$$

has a radius of convergence $R = 2$ (so we know that this series converges at least on the open interval $(-1, 3)$.) Find the *interval of convergence* for this series. Show step-by-step work.

To find the interval of convergence we test the endpoints of the open interval of convergence.

- Test $x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3-1)^n}{2^n n^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^3}$,
  which converges by the alternating series test (or by the fact that the series of absolute values, namely $\sum 1/n^3$ converges by the integral test.)

- Test $x = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (-1-1)^n}{2^n n^3} = \sum_{n=0}^{\infty} \frac{1}{n^3}$, which converges by the integral test.

So the interval of convergence is $[-1, 3]$.

NOTE: observe that in its current form, this series is undefined at $n = 0$. If you noticed so explicitly in the exam and solved the problem accordingly, you received the appropriate credit. The solution just described is correct if the series started at, $n = 1$ for instance.