6. (10 points) A quantity has density function $p(x)$, where

$$p(x) \quad = \quad \begin{cases} 0 & \text{when } x < 0, \\ a + bx^2 & \text{when } 0 \le x \le 1, \\ 0 & \text{when } x > 1. \end{cases}$$

Assuming that the mean value of the quantity is $\frac{3}{4}$, find a and b. Show your work.

We know

$$1 = \int_0^1 (a + bx^2) \, dx = \left(ax + b \frac{x^3}{3} \right) \Big|_0^1 = a + \frac{b}{3},$$

$$3/4 = \int_0^1 (ax + bx^3) \, dx = \left(a \frac{x^2}{2} + b \frac{x^4}{4} \right) \Big|_0^1 = \frac{a}{2} + \frac{b}{4}.$$

The first equation tells us that $a = 1 - b/3$. Substituting this into the second equation, yields $b/12 = 1/4$ or $b = 3$. Substituting this value into the first equation tells us that $a = 0$.

Thus, $a = 0$ and $b = 3$.