4. [12 points] The following three parts of this problem have to do with the convergence of series.

(a) [4 points of 12] For $\Sigma \frac{5}{1+n+e^n}$:

i. [2 points of 4] What is a good test to determine the convergence of this series? Explain, in 1–2 sentences only, why this is.

Solution:
The Comparison Test is a good test for this series. This is because we know the convergence of $\Sigma \frac{5}{e^n}$, and note that $\frac{5}{1+n+e^n} < \frac{5}{e^n}$, so we have a ready made comparison to test the convergence of $\Sigma \frac{5}{1+n+e^n}$. Limit comparison will also work (which is reasonable because there's an obvious comparison series), as will the ratio test (which is reasonable because of the exponential term).

ii. [2 points of 4] Determine if this series converges, diverges, or if we can't tell.

Solution:
Because $\frac{5}{1+n+e^n} < \frac{5}{e^n}$, and because we know that $\Sigma \frac{5}{1+n+e^n}$ converges, we know that $\Sigma \frac{5}{1+n+e^n}$ must also converge. Alternately, limit comparison with the same two series gives $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^n}{1+n+e^n} = 1$, a finite non-zero number, so both must converge. And also alternately, the ratio test gives $\lim_{n\to\infty} |\frac{1+n+e^n}{1+n+e^{n+1}}| = \frac{1}{e} < 1$, so, again, the series must converge. Math is astonishingly consistent.

(b) [4 points of 12] For $\Sigma \frac{n}{n^2+5}$:

i. [2 points of 4] What is a good test to determine the convergence of this series? Explain, in 1–2 sentences only, why this is.

Solution:
The Integral Test is a good test for this series. This is because it is easy to integrate $\frac{x}{x^2+5}$, so the integral test is a dandy choice. The limit comparison test is another good option, because $\frac{n}{n^2+5} \sim \frac{n}{n^2} = \frac{1}{n}$ when $n \to \infty$, which makes us think this series must diverge, but because $\frac{n}{n^2+5} < \frac{n}{n^2}$ the comparison test doesn't work when comparing to $\frac{1}{n}$. It's possible to use the comparison test, but that requires quite a bit of cunning to use correctly.

ii. [2 points of 4] Determine if this series converges, diverges, or if we can't tell.

Solution:
Integrating, $\int_1^\infty \frac{x}{x^2+5}\,dx = \lim_{b\to\infty} \frac{1}{2}\left(\ln(b^2+5) - \ln(6)\right)$, which diverges as $b \to \infty$, so by the integral test the series $\Sigma \frac{n}{n^2+5}$ must also diverge. Alternately, using the limit comparison test with the two series $\Sigma \frac{n}{n^2+5}$ and $\Sigma \frac{1}{n}$, we have $\lim_{n\to\infty} \frac{n}{n^2+5} \cdot \frac{n}{1} = 1$, so that, knowing that $\Sigma \frac{1}{n}$ diverges, we must have that our series diverges.

this problem is continued on the following page...