6. [16 points] Chris has decided to take flying lessons, and notices that the cross-section of the airplane wing is given approximately by the figure to the right. The front-to-back length of the wing, as shown in the figure, is 2 m. The end-to-end length of the wing is 15 m (that is, its length along an axis coming out of this page is 15 m), and its ends are flat.

(a) [3 points of 16] If this cross-section is described by the polar equation \( r = a \cos(3\theta) \), what is \( a \)?

Solution:
The indicated front-to-back length occurs when \( \theta = 0 \). \( r(0) = a \), so we must have \( a = 2 \) m.

(b) [4 points of 16] What range of values for \( \theta \) generate this figure?

Solution:
The curve starts and ends at \( r = 0 \), which requires that \( \cos(3\theta) = 0 \), so that \( 3\theta = \pm \frac{\pi}{2} \) is a good solution. Thus \( \frac{-\pi}{6} \leq \theta \leq \frac{\pi}{6} \). There are many other sets of \( \theta \) values that give the region as well (e.g., any interval that gives the top half of the wing, e.g., \([0, \frac{\pi}{6}]\), \([\frac{2\pi}{3}, \frac{5\pi}{6}]\), \([\frac{4\pi}{3}, \frac{3\pi}{2}]\), \([2\pi, \frac{13\pi}{6}]\), etc., plus any interval that gives the bottom half, e.g., \([-\frac{\pi}{6}, 0]\), \([\frac{\pi}{2}, \frac{2\pi}{3}]\), \([\frac{5\pi}{6}, \frac{4\pi}{3}]\), \([\frac{13\pi}{6}, 2\pi]\), etc.).

(c) [9 points of 16] Airplanes frequently have fuel tanks in their wings. If 75% of the wing’s volume is available space for a fuel tank, what volume of fuel could be stored in this wing?

Solution:
The volume of the wing is given by the area of its cross-section times 15 m. Slicing with polar slices, a slice of the cross-sectional area is given by \( \Delta A \approx \frac{1}{2} r^2 \Delta \theta = \frac{1}{2} (2 \cos(3\theta))^2 \Delta \theta \), so that the total cross-sectional area is given by \( \int_{-\pi/6}^{\pi/6} 2 \cos^2(3\theta) \, d\theta \). Thus the total volume is \( 5\pi \) m\(^3\), and the total volume available for fuel storage is \( \frac{15\pi}{4} \) m\(^3\), or about 11.8 m\(^3\).