1. [6 points] Use the integral test to determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n^5)}$$

converges or diverges. Explain your reasoning.

Solution:
The integral test says that if $f(n) > 0$ and $f'(n) < 0$, then the series $\sum_{n=a}^{\infty} f(n)$ converges or diverges $\int_c^\infty f(n)\,dn$ converges of diverges. Here, $f(n) = \frac{1}{n\ln(n^5)}$ is positive and decreasing, so we can investigate the convergence of the integral. Noting that $\ln(n^5) = 5\ln(n)$ and using the substitution $w = \ln(n)$, we have

$$\int_c^\infty \frac{1}{n\ln(n^5)}\,dn = \int_c^\infty \frac{1}{5n\ln(n)}\,dn = \frac{1}{5}\int_{\ln(c)}^\infty \frac{1}{w}\,dw,$$

which we know is divergent. Thus the series must diverge as well. Note that if we hadn't remembered that $\ln(n^5) = 5\ln(n)$, we could take $w = \ln(n^5)$ so that $w' = \frac{5n^4}{n^5}$ and $\frac{1}{5}dw = \frac{1}{n}dn$. Thus, we get the integral $\int_{\ln(c^5)}^\infty \frac{1}{5w\ln(w)}\,dw$, which leads to the same conclusion as before.

2. [6 points] Use one of the ratio or limit comparison tests to determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3^n - 2}}$$

converges. Explain your reasoning.

Solution:
With the ratio test, taking $a_n = |a_n| = \frac{1}{\sqrt{3^n-2}}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{3^n-2}}{\sqrt{3^{n+1}-2}} = \frac{\sqrt{3^n}\sqrt{1-\frac{2}{3^n}}}{\sqrt{3^n}\sqrt{3-\frac{2}{3^n}}}$. Thus, canceling the factor of $\sqrt{3^n}$, we have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\sqrt{1-\frac{2}{3^n}}}{\sqrt{3-\frac{2}{3^n}}} = \frac{1}{\sqrt{3}}$. This is less than one, so by the ratio test the series must converge.

Alternately, to use the limit comparison test, we need a good comparison series. For large n, $a_n = \frac{1}{\sqrt{3^n-2}}$ looks like $b_n = \frac{1}{\sqrt{3^n}} = \frac{1}{3^{n/2}}$, and we know that $\sum_{n=1}^{\infty} b_n$ converges (because it's a geometric series with $x < 1$). Using this for our comparison, we have $\frac{a_n}{b_n} = \frac{\sqrt{3^n}}{\sqrt{3^n-2}}$, so that, by factoring out $\sqrt{3^n}$ as above, as $n \to \infty$ we have $\frac{a_n}{b_n} \to 1$, and therefore $\sum a_n$ must also converge.