6. [12 points] Recall that the Great Pyramid of Giza was (originally) approximately 480 ft high, with a square base approximately 760 ft to a side. The pyramid was made of close to 2.4 million limestone blocks, and has several chambers and halls that extended into its center. It is not too far from the truth to suppose that these open areas are located along the vertical centerline of the pyramid, and that we can therefore think of the density of the pyramid varying only along its vertical dimension. Suppose that the result is that the density of the pyramid is approximately $\delta(h) = (0.00011(h-240)^2 + 134.2)$ lb/ft^3, where h is the height measured up from the base of the pyramid.

(a) [6 points of 12] Set up an integral to find the weight W of the pyramid. You need not evaluate the integral to find the actual weight.

Solution:
Because the density of the pyramid varies along its vertical dimension, we have to slice it in that direction to find the weight of each slice and then sum them with an integral. If z is the distance down from the top of the pyramid and x is the length of the edge of the square slice, the volume of the slice is $\Delta V = x^2\Delta z$. From similar triangles, we have that $\frac{x}{z} = \frac{760}{480}$, so that $x = \frac{760}{480}z$, so that $\Delta V = (\frac{760}{480})^2 z^2 \Delta z$. By symmetry we can see that $\delta(h) = \delta(z)$, or we can find this explicitly: $\delta(h) = \delta(480 - z) = 0.00011(480 - z - 240)^2 + 134.2 = 0.00011(240 - z)^2 + 134.2 = 0.0001(z - 240)^2 + 134.2 = \delta(z)$. Thus the weight of the pyramid is

$$W = \int_0^{480} \left(\frac{760}{480}\right)^2 z^2 (0.0001(z-240)^2 + 134.2)\, dz.$$

Evaluating this numerically, we can find $W \approx 1.2615 \times 10^{10}$ lbs (about 12.6 billion pounds).

Alternately, in terms of h, the distance up from the ground, $\Delta V = (760 - \frac{760}{480}h)^2 \Delta h$, so that $W = \int_0^{480} (760 - \frac{760}{480}h)^2 (0.00011(h-240)^2 + 134.2)\, dh$.

(b) [6 points of 12] Give an expression, in terms of integral(s), that tells how far off the ground the center of mass of the pyramid is. Again, you need not evaluate the integral(s). *(Note that you may set up the expression in terms of the weight density without worrying about converting it to a mass density.)*

Solution:
Again working with z, the distance down from the top of the pyramid, we have

$$\overline{z} = \frac{\int_0^{480} \left(\frac{760}{480}\right)^2 z^2 \delta(z) \cdot z\, dz}{W} = \frac{\int_0^{480} \left(\frac{760}{480}\right)^2 z^3 (0.0001(z-240)^2 + 134.2)\, dz}{W},$$

where W is given in part (a). Then the height of the center of mass above the surface of the desert is $\overline{h} = 480 - \overline{z}$. Using $W = 1.2615 \times 10^{10}$ lbs, found in (a), we can find $\overline{z}$ and $\overline{h}$ by evaluating the integrals numerically: $\overline{z} \approx 361$ ft, so $\overline{h} \approx 480 - \overline{z} = 119$ feet above the ground.