9. [15 points] Suppose that we know that $\sum_{n=1}^{\infty} a_n$ converges, *and that* $|a_{n+1}| < |a_n|$—but we don't know what a_n is. For each of the series below, determine whether it converges, diverges, or we cannot tell (that is, there could be one value for a_n that would converge and one that would not). Circle your answer and provide a short but careful explanation for your answer (how do we know the series converges or diverges?, or, what examples show that we cannot tell?). *(The italicized instruction was omitted from the printed version of the exam.)*

(a) [3 points of 15] $\sum_{n=1}^{\infty} |a_n|$ converges diverges cannot tell

Solution:
We cannot tell if this converges or not. For example, if $a_n = (-1)^n/n$ then the original series converges, but this series does not. But if $a_n = 1/n^2$, then both converge.

(b) [3 points of 15] $\sum_{n=1}^{\infty} (-1)^n |a_n|$ converges diverges cannot tell

Solution:
Because we know that $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} a_n = 0$ (because $\sum a_n$ converges), we have convergence by the alternating series test.

(c) [3 points of 15] $\sum_{n=1}^{\infty} \frac{a_n+1}{a_n+5}$ converges diverges cannot tell

Solution:
This diverges. We know that $\lim_{n\to\infty} a_n = 0$, so the terms of this series do not go to zero: $\lim_{n\to\infty} \frac{a_n+1}{a_n+5} = \frac{1}{5}$. Thus this series must diverge.

(d) [3 points of 15] $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges diverges cannot tell

Solution:
This converges. Clearly $|\frac{a_n}{n^2}| < \frac{1}{n^2}$ for large enough n, so we know that $\sum \frac{a_n}{n^2}$ converges absolutely.

(e) [3 points of 15] $\sum_{n=1}^{\infty} \frac{3^n a_n}{n^3}$ converges diverges cannot tell

Solution:
We cannot tell if this converges or not. For example, if $a_n = 1/n^2$, this series is $\sum \frac{3^n}{n^5}$, which diverges because exponentials dominate power functions and the terms in the series therefore do not go to zero. However, if $a_n = 1/3^n$, it clearly converges.