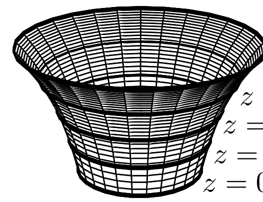
4. [16 points] An entrepreneurial University of Michigan Business Squirrel is marketing childrens' buckets with curved sides, as shown in the figure to the right, below. The figure gives the radius of the bucket, r, at different heights, z, from the bottom of the bucket. All lengths are given in inches. Suppose that a child fills one of these buckets with muddy water.



a. [4 points] If the density of the water in the bucket is $\delta(z)$ oz/in^3, write an integral that gives the mass of the water in the bucket.

Solution: Slicing the bucket horizontally, the mass is

$$M = \int_0^8 \pi\,\delta(z)\,(r(z))^2\,dz \quad \text{oz.}$$

b. [4 points] If $\delta(z) = (24 - z)$ oz/in^3, estimate the mass using your integral from (a).

Solution: The data given provide values of $r(z)$ at steps of size $\Delta z = 2$ in. We can find $\delta(z)$ at each of the points $z = 0$, 2, etc., and find a left or right Riemann sum for the mass. A better estimate would be the average of the two (the trapezoid estimate):

$$\begin{aligned}\text{LEFT} &= 2\pi\left(24(1)^2 + 22(3/2)^2 + 20(9/4)^2 + 18(3^2)\right) \approx 2116 \text{ oz}\\ \text{RIGHT} &= 2\pi\left(22(3/2)^2 + 20(9/4)^2 + 18(3)^2 + 16(6^2)\right) \approx 5584 \text{ oz}\\ \text{TRAP} &\approx 0.5\,(2116 + 5584) = 3850 \text{ oz}\end{aligned}$$

c. [8 points] Estimate the center of mass of the bucket.

Solution: By symmetry, the center of mass must be along the centerline of the bucket, so if $\overline{x}$ and $\overline{y}$ give the coordinates along the base of the bucket, we know $\overline{x} = 0$ and $\overline{y} = 0$. The z center of mass is

$$\overline{z} = \frac{\int_0^8 \pi\, z\,\delta(z)\,(r(z))^2\,dz}{M} \quad \text{in,}$$

where M is the mass given in (a). We have the mass from (b), and so need only estimate the integral $\int_0^8 \pi\, z\,\delta(z)\,(r(z))^2\,dz$. Again, we can find a left or right Riemann sum or a trapezoid estimate:

$$\begin{aligned}\text{LEFT} &= 2\pi\left(24(0)(1)^2 + 22(2)(3/2)^2 + 20(4)(9/4)^2 + 18(6)(3)^2\right) = 2952\pi \approx 9274\\ \text{RIGHT} &= 2\pi\left(22(2)(3/2)^2 + 20(4)(9/4)^2 + 18(6)(3)^2 + 16(8)(6)^2\right) = 12,168\pi \approx 38,227\\ \text{TRAP} &\approx 0.5\,(2952\pi + 12,168\pi) = 7560\pi \approx 23,750.\end{aligned}$$

Then the center of mass is $\overline{z} \approx 23,750/3850 = 6.17$ in. Using just a left-sum, we have $\overline{z} \approx 9274/2116 = 4.60$ in, and the right-sum gives $\overline{z} \approx 38,227/5584 = 6.85$ in.