5. [13 points] Suppose that when a fire alarm is set off in East Hall, the occupants (being mathematicians) leave at precise five-minute intervals. At the end of each interval, 75% of those who were in the building at the beginning of the interval exit the building. Suppose that on a sunny Friday afternoon at 2PM a fire alarm goes off when there are 400 mathematicians in East Hall.

a. [4 points] Find the number of mathematicians that leave at the end of the first, second, and nth five-minute intervals.

Solution: It is easiest to start with the general case: the number of mathematicians in the building at the beginning of the nth five-minute time interval is $400(0.25)^{n-1}$. Thus the number of mathematicians leaving at the end of the nth five minute interval is $0.75(400(0.25)^{n-1}) = 300(0.25)^{n-1}$. This means that the number leaving at the end of the first five-minute interval is 300 ($= 300(0.25)^0$), and the second, 75 ($= 300(0.25)^1$).

b. [5 points] Let $L(n)$ be the total number of mathematicians who have left East Hall at the end of the nth five-minute interval after the alarm started. Find a closed-form expression for $L(n)$.

Solution: We note that $L(n)$ is just the sum of the terms in (a):

$$L(n) = 300 + 300(0.25) + \cdots + 300(0.25)^{n-1}.$$

We note that this is just a finite geometric series with n terms and a coefficient of 300. Thu

$$L(n) = 300\left(\frac{1-0.25^n}{1-0.25}\right) = 400\left(1-0.25^n\right).$$

(Alternately, we could note that at the end of the nth five-minute interval the number of mathematicians left in the building is $400(0.25)^n$. Thus, the number who have left must be $400 - 400(0.25)^n = 400(1-0.25^n)$.)

c. [4 points] How many mathematicians will leave the building if the alarm goes on forever? (Justify your answer mathematically.)

Solution: As $n \to \infty$, the numerator of $L(n)$ goes to one, so $L(n) \to 300\frac{1}{1-0.25} = 400$. So all of the mathematicians actually leave!