6. [15 points] For each of the following, assume that $\sum a_n$ and $\sum b_n$ are both convergent series, and that $a_n > a_{n+1} > b_n > b_{n+1} > 0$. For each, explain your answer in a sentence or two, or with a clear picture or counterexample.

a. [3 points] Is $\sum (b_n - a_n)$ a convergent series? Explain.

Solution: Yes, $\sum (b_n - a_n)$ is a convergent series. We know $\sum a_n$ and $\sum b_n$ converge, so $\sum (b_n - a_n)$ must converge to $\sum b_n - \sum a_n$.

b. [3 points] Is $\sum (a_n \cdot b_n)$ a convergent series? Explain.

Solution: Yes, $\sum a_n \cdot b_n$ is convergent. We know that $a_n \to 0$ as $n \to \infty$, so for some sufficiently large value of n, $a_n < 1$. Thus for sufficiently large values of n, $a_n \cdot b_n < b_n$, and therefore because $\sum b_n$ converges, by the comparison test $\sum a_n \cdot b_n$ must also converge.

c. [3 points] Is $\sum ((-1)^n \ln(a_n + 1))$ a convergent series? Explain.

Solution: Yes, $\sum (-1)^n \ln(a_n + 1)$ is a convergent series. We know (because $\sum a_n$ converges) that $a_n \to 0$ as $n \to \infty$, so $\ln(a_n + 1) \to \ln(1) = 0$, and we're given that $a_n > a_{n+1} > 0$, so $\ln(a_n + 1) > \ln(a_{n+1} + 1) > 0$. Thus $\ln(a_n + 1) \to 0$ monotonically from above as $n \to \infty$, and $\sum (-1)^n \ln(a_n + 1)$ is an alternating series. Thus, by the alternating series test we know that the series converges.

d. [3 points] Is $\sum (2\, a_n)$ a convergent series? Explain.

Solution: Yes, $\sum 2\, a_n$ is a convergent series. Because $\sum a_n$ is convergent, $2 \sum a_n$ is clearly well-defined, and $2 \sum a_n = \sum 2\, a_n$.

e. [3 points] Is $\sum ((-1)^n \sqrt{b_n}\,)$ an absolutely convergent series? Explain.

Solution: We can't tell if $\sum (-1)^n \sqrt{b_n}$ is absolutely convergent. This requires that $\sum \sqrt{b_n}$ converge, which we can't tell because for sufficiently large n it must be that $\sqrt{b_n} > b_n$, and thus the convergence of $\sum b_n$ doesn't tell us what happens to $\sum \sqrt{b_n}$.