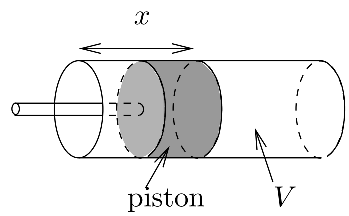
7. [15 points] Consider a piston that compresses a closed cylinder of gas, as shown in the figure to the right, below. If the volume of the gas in the cylinder is V, then the force required to move the piston and compress the gas is $F = \frac{k}{V^{1.4}}$, where k is a constant. The uncompressed length of the gas cylinder is 2 ft and its radius is $\frac{1}{4}$ ft. Let x be the distance that the piston has moved to compress the gas. (Note that the volume of a cylinder with radius r and height h is $\pi r^2 h$.)

a. [5 points] Find an expression for $F(x)$, the force as a function of x. If $F(0) = 200$ lb, find k.



Solution: The volume when the gas has been compressed a distance x is $V = \pi\,(\frac{1}{4})^2\,(2-x)$, so the force as a function of x is $F(x) = \frac{k}{((\pi/16)\,(2-x))^{1.4}}$. When $x = 0$ the volume is $V = \pi\,(\frac{1}{4})^2\,2 = \frac{\pi}{8}$ ft^3, so we have $F(0) = 200 = \frac{k}{(\pi/8)^{1.4}}$, and $k = 200(\frac{\pi}{8})^{1.4} \approx 54$.

b. [10 points] Find the work to compress the gas from $x = 0$ to $x = \frac{3}{2}$.

Solution: Using $F(x)$ from above, the work to compress the gas from a distance x to $x + \Delta x$ is $\Delta W = \frac{k}{((\pi/16)\,(2-x))^{1.4}}\,\Delta x$. Thus, letting $\Delta x \to 0$, the total work to compress the gas from $x = 0$ to $x = \frac{3}{2}$ is

$$\int_0^{3/2} \frac{k}{((\pi/16)\,(2-x))^{1.4}}\,dx.$$

Integrating, we have

$$\int_0^{3/2} \frac{k}{((\pi/16)\,(2-x))^{1.4}}\,dx = \frac{k}{0.4(\pi/16)^{1.4}}\,(2-x)^{-0.4}\Big|_0^{3/2}$$
$$= \frac{k}{0.4(\pi/16)^{1.4}}\,((\frac{1}{2})^{-0.4} - 2^{-0.4}) \approx 741 \text{ ft} \cdot \text{lb}.$$