8. [16 points] For each of the following series, state a convergence test that you could use to determine if the series converges or not and indicate why you chose that test. Then carefully apply the test to determine if the series converges or not. Mathematical precision is important in this problem.

a. [8 points] $\sum_{n=2} \frac{\sqrt{n+3}}{n^2-1}$

Solution: This looks like a good problem to try and do a comparison test, but the most obvious comparison ($\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$) isn't larger than the given function, so we use the *Limit Comparison Test*. We know that $\sum n^{-3/2}$ converges. The limit we consider is $\lim_{n\to\infty} \frac{a_n}{b_n}$, which with $a_n = \frac{\sqrt{n+3}}{n^2-1}$ and $b_n = \frac{1}{n^{3/2}}$ is

$$\lim_{n\to\infty} \frac{\sqrt{n+3}}{n^2-1} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^2\sqrt{1+\frac{3}{n}}}{n^2-1} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{3}{n}}}{1-\frac{1}{n^2}} = 1.$$

This is a finite non-zero limit, so we know that the convergence properties of both of these series are the same, so the given series must converge.

b. [8 points] $\sum \frac{(n+1)!}{2e^{3n}}$

Solution: This problem involves factorials and exponentials, so the ratio test is a good one to use. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{(n+2)!}{(2\,e^{3n+3})!}\right)\left(\frac{2e^{3n}}{(n+1)!}\right) = \lim_{n\to\infty} \left(\frac{n+2}{e^3}\right) \to \infty.$$

This diverges, so by the ratio test the series diverges.