4. [14 points] Using a rope, it takes 8 minutes to lift a 50-pound box of dirt from the ground to a height of 20 feet above the ground. As the box is lifted, dirt falls out of the box at a constant rate such that a total of 2 pounds of dirt has been lost when it reaches the final height of 20 feet.

a. [4 points] Let h be the height, in feet, of the box above the ground. Write an expression estimating the work done in raising the box from a height of h_i feet to a height of $h_i + \Delta h$ feet, ignoring the weight of the rope.

Solution: The change in weight will be $-\frac{1}{10}$ pound per foot lifted. The weight of the box after it has been lifted h_i feet off the ground is $50 - \frac{1}{10}h_i$. When the box is lifted from h_i feet to a height of $h_i + \Delta h$ feet, it is lifted Δh feet. The work done to lift the box this distance is then Work $\approx (50 - \frac{1}{10}h_i)\Delta h$ foot-pounds.

b. [3 points] Find the total amount of work done in raising the box of dirt from the ground to the final height of 20 feet, ignoring the weight of the rope.

Solution: The total work done is given by the definite integral

$$\text{Work} = \int_0^{20} (50 - \frac{1}{10}h_i)dh = 50h - \frac{1}{20}h^2|_0^{20} = 980$$

The total work in raising the box of dirt is 980 foot-pounds.

c. [7 points] Suppose the rope lifting the box of dirt weighs 1.5 pounds per foot and dangles from a platform that is 30 feet above the ground. Find the total work done to lift the box of dirt to a height of 20 feet above the ground, taking into account the weight of the rope.

Solution: Now we will find the work done to lift the rope. The bottom ten feet of rope (ten feet of rope closest to the box) will move the full 20 feet. The weight of this section of rope is $(1.5)(10) = 15$ pounds, so the work done on this part of the rope is $(15)(20) = 300$ foot-pounds.
The top 20 feet of the rope will not move the full 20 feet. Let y measure the distance from the top of the platform to a small slice of rope of size Δy. The weight of the small slice is $1.5\Delta y$ pounds. A piece that is y_i feet from the top moves y_i feet as the box is lifted. The work done to lift the small slice of rope is then $(1.5\Delta y)(y_i)$ foot-pounds. We can sum up all such slices, which leads to the definite integral

$$\text{Work} = \int_0^{20} 1.5y dy = 0.75y^2|_0^{20} = 300$$

So 300 foot-pounds of work are done to lift this part of the rope. The total work to lift the rope is 600 foot-pounds. The total work done to lift the box, including both the work done on the box and on the rope, is then 1580 foot-pounds.