**7**. [16 points] Consider the following two functions defined by power series:

$$f(x) = \sum_{n=0}^{\infty} a_n(x+1)^n \text{ and } g(x) = \sum_{n=0}^{\infty} b_n(x-2)^n,$$

where $a_n, b_n > 0$ for every $n$, and $a_n \to 0$ as $n \to \infty$. Both $f(x)$ and $g(x)$ converge at $x = -2$, but $f(x)$ diverges at $x = 0$ and $g(x)$ diverges at $x = 7$.

**a**. [4 points] What is the interval of convergence for the function $f(x)$? Justify your answer using complete sentences.

*Solution:* The interval of convergence is $-2 \le x < 0$. Since $f(x)$ is based at $x = -1$, converges at $x = -2$, and diverges at $x = 0$, we know the radius of convergence is exactly one, and we know which endpoints are included in the interval.

**b**. [4 points] Give lower and upper bounds for the radius of convergence for the function $g(x)$. Justify your answer using complete sentences.

*Solution:* Similarly, $g(x)$ converges at $x = -2$ and is based at 2, so its radius of convergence is at least 4. However, since $g(x)$ diverges at $x = 7$, it has a radius of convergence of at most 5. So the radius of convergence of $g(x)$ is at least 4 and at most 5.

**c**. [4 points] Write the series $\sum_{n=0}^{\infty} \left(b_n + (-1)^n \frac{a_n}{2^n}\right)$ in terms of $f$ and $g$ to determine if this series converges, diverges, or it is impossible to tell. Give a brief explanation of your answer using complete sentences.

*Solution:* The series in question converges, since we can write the above series as

$$\sum_{n=0}^{\infty} b_n + (-1)^n \frac{a_n}{2^n} = \sum_{n=0}^{\infty} b_n(1)^n + a_n(-1/2)^n = g(3) + f(-3/2).$$

We know both $g(3)$ and $f(-3/2)$ converge because the inputs are within the radius of convergence for each function. Therefore the series converges.

**d**. [4 points] Suppose we also know that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \frac{1}{5}$. What is the radius of convergence of the power series $h(x) = \sum_{n=0}^{\infty} b_{2n+1}x^n$? *(Hint: You might find it helpful to use the fact that* $\frac{a}{c} = \left(\frac{a}{b}\right)\left(\frac{b}{c}\right)$*.)*

*Solution:* Using the Ratio test, we have

$$\begin{aligned} \lim_{n\to\infty} |b_{2(n+1)+1}x^{n+1}/b_{2n+1}x^n| &= \lim_{n\to\infty} |b_{2n+3}x/b_{2n+1}||x| \\ &= \lim_{n\to\infty} |b_{2n+3}/b_{2n+2}| \cdot |b_{2n+2}/b_{2n+1}||x| \\ &= 1/25|x| \end{aligned}$$

Therefore the radius of convergence is 25.