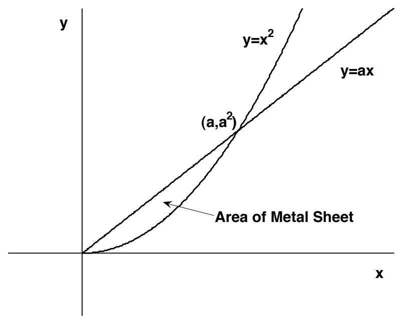
8. [13 points] Consider a solid metal sheet bounded by the curves $y = x^2$ and $y = ax$, for constant $a > 0$. The density of the sheet is given by $\delta(x) = 4$ grams per square centimeter.

a. [3 points] Sketch the area of the metal sheet in the space provided below. Be sure to label your graphs and axes.

Solution:



b. [4 points] Find the exact mass of the sheet, and be sure to include appropriate units. Your answer may be in terms of a.

Solution: Since the density is constant throughout the region, mass is the product of the area of the sheet and its density.

$$\text{Mass} = 4\int_0^a (ax - x^2)dx = 4\left(\frac{a}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^a = 2a^3 - \frac{4a^3}{3} = \frac{2a^3}{3}$$

The mass of the sheet is $\frac{2a^3}{3}$ grams.

c. [3 points] Find $\bar{x}$, the x-coordinate for the center of mass. Your answer may be in terms of a.

Solution:

$$\text{Moment} = \int_0^a 4x(ax - x^2)dx = \int_0^a (4ax^2 - 4x^3)dx = \left(\frac{4ax^3}{3} - x^4\right)\Big|_0^a = \frac{4}{3}a^4 - a^4 = \frac{1}{3}a^4$$

So $\bar{x} = \frac{\frac{a^4}{3}}{\frac{2a^3}{3}} = \frac{1}{2}a$.

d. [3 points] Find $\bar{y}$, the y-coordinate for the center of mass. Your answer may be in terms of a.

Solution:

$$\text{Moment} = \int_0^{a^2} 4y(\sqrt{y} - \frac{y}{a})dy = \int_0^{a^2}\left(4y^{3/2} - \frac{4}{a}y^2\right)dy = \left(\frac{8}{5}y^{5/2} - \frac{4}{3a}y^3\right)\Big|_0^{a^2} = \frac{4}{15}a^5$$

So $\bar{y} = \frac{\frac{4}{15}a^5}{\frac{2}{3}a^3} = \frac{2}{5}a^2$.