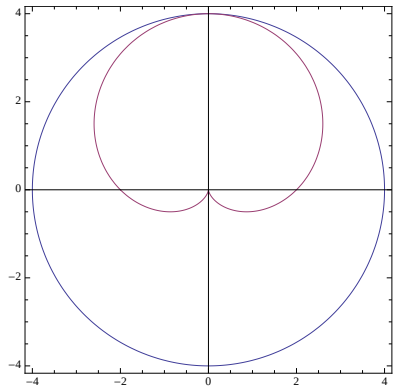
2. [14 points] The graph of the circle $r = 4$ and and the cardioid $r = 2\sin\theta - 2$ are shown below.



a. [3 points] Write a formula for the area inside the circle and outside the cardioid in the first quadrant.

Solution: Area of the quarter of a circle$= 4\pi$
Area of cardioid$=\int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}(2\sin\theta - 2)^2 d\theta$
Area$=4\pi - \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}(2\sin\theta - 2)^2 d\theta$

b. [7 points] At what angles $0 \le \theta < 2\pi$ is the minimum value of the y coordinate on the cardioid attained? No credit will be given for answers without proper mathematical justification.

Solution:
$y(\theta) = (2\sin\theta - 2)\sin\theta$

$y'(\theta) = 2\cos\theta\sin\theta + (2\sin\theta - 2)\cos\theta = 4\cos\theta\sin\theta - 2\cos\theta$

Critical points $\quad (4\sin\theta - 2)\cos\theta = 0$

$\cos\theta = 0 \qquad$ or $\qquad \sin\theta = \frac{1}{2} \qquad$ then $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.
Minimum y coordinate at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

c. [4 points] Write an integral that computes the value of the length of the piece of the cardioid lying below the x-axis.

Solution:
$x(\theta) = (2\sin\theta - 2)\cos\theta \qquad x'(\theta) = 2\cos^2\theta - (2\sin\theta - 2)\sin\theta$

$L = \int_0^{\pi} \sqrt{(2\cos^2\theta - (2\sin\theta - 2)\sin\theta)^2 + (4\cos\theta\sin\theta - 2\cos\theta)^2}\, d\theta$