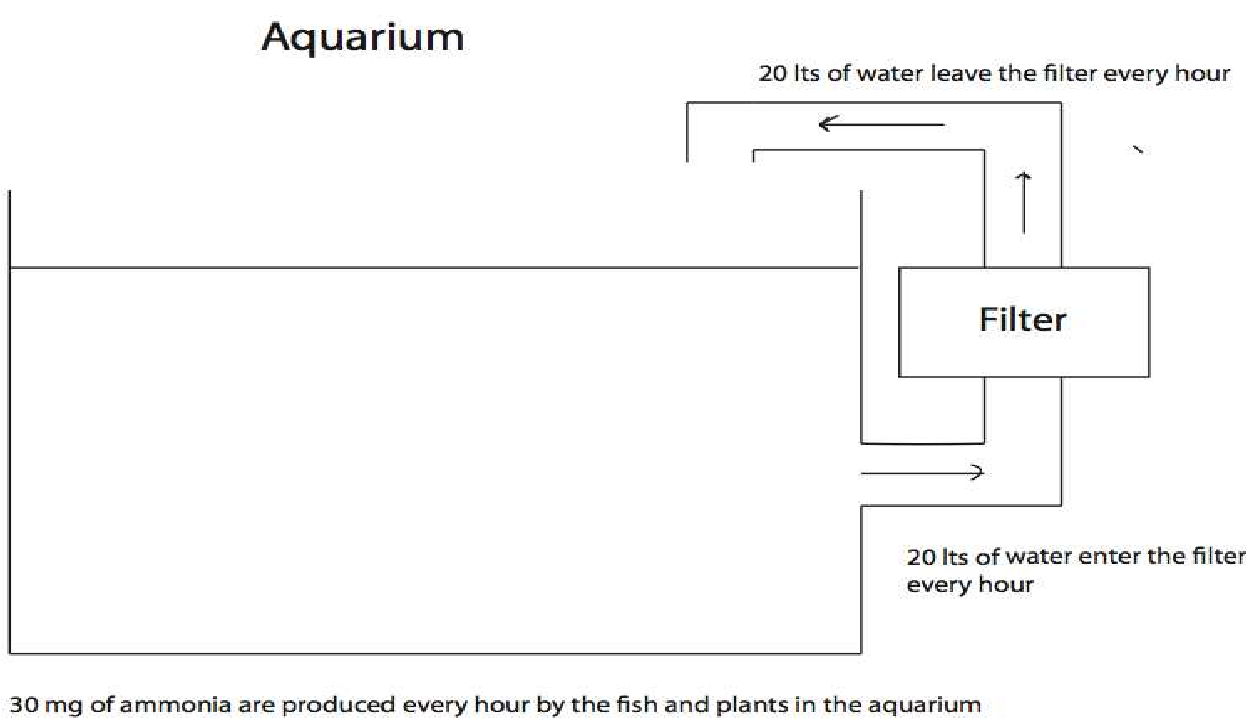
7. [15 points] An aquarium containing 100 liters of fresh water will be filled with a variety of small fish and aquatic plants.

A water filter is installed on the tank to help remove the ammonia produced by the decomposing organic matter generated by plants and fish in the aquarium. The filter takes water from the tank at a rate of 20 liters every hour. The water then is filtered and returned to the aquarium at the same rate of 20 liters every hour. Ninety percent of the ammonia contained in the water that goes through the filter is removed.

It is estimated that the fish and plants produce 30 mg of ammonia every hour. Assume the ammonia mixes instantly with the water in the aquarium.



a. [6 points] Let $Q(t)$ be the amount in mg of ammonia in the fish tank t hours after the fish were introduced into the aquarium. Find the differential equation satisfied by $Q(t)$. Include its initial condition.

Solution:

$$\frac{dQ}{dt} = 30 - 20\left(\frac{Q}{100}\right) + (.1)20\left(\frac{Q}{100}\right)$$
$$\frac{dQ}{dt} = 30 - .18Q \qquad Q(0) = 0$$

b. [7 points] Find the amount of ammonia in the fish tank 3 hours after the fish were introduced into the aquarium. Make sure to include units in your answer.

Solution:

$$\frac{dQ}{dt} = 30 - .18Q$$
$$\frac{dQ}{30 - .18Q} = dt$$
$$-\frac{1}{.18}\ln|30 - .18Q| = t + C$$
$$Q(t) = \frac{30}{.18} + De^{-.18t} = 166.6 + De^{-.18t}$$
$$\text{using } Q(0) = 0 \text{ you get} \qquad Q(t) = 166.6(1 - e^{-.18t})$$
$$Q(3) = 69.53 \text{ mg}$$

c. [2 points] What happens to the value of $Q(t)$ in the long run?

Solution:

$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 166.6(1 - e^{-.18t}) = 166.6.$

The value of $Q(t)$ converges to 166.6 mg.