2. [12 points] Consider a particle whose trajectory in the xy-plane is given by the parametric curve defined by the equations

$$x(t) = t^4 - 4t^2, \qquad y(t) = t^2 - 2t,$$

for $-3 \leq t \leq 3$. Show all your work to receive full credit.

a. [3 points] Is there any value of t at which the particle ever comes to a stop? Justify.

Solution: No. For the particle to come a stop, its velocity in both the x- and y-direction must be zero. We have that

$$\frac{dx}{dt} = 4t^3 - 8t = 4t(t^2 - 2) = 0$$

at $t = 0, \pm\sqrt{2}$ and

$$\frac{dy}{dt} = 2t - 2 = 0$$

at $t = 1$. Since there are no times at which $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are simultaneously zero, the particle never comes to a stop.

b. [2 points] For what values of t does the path of the particle have a vertical tangent line?

Solution: Vertical tangent lines occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. From the above calculation, this is true at $t = 0, \pm\sqrt{2}$.

c. [3 points] What is the lowest point (x, y) on the curve?

Solution: We want to minimize the value of the y-coordinate over $-3 \leq t \leq 3$. The only critical point for $y(t)$ was found above at $t = 1$. Since $\left.\frac{dy}{dt}\right|_{t=0} = -2 < 0$ and $\left.\frac{dy}{dt}\right|_{t=2} = 2 > 0$, the First Derivative Test tells us that $t = 1$ is a local minimum, and thus a global minimum since it is the only critical point on the given interval. The lowest point on the curve is thus $(x(1), y(1)) = (-3, -1)$.

d. [2 points] At what values of t does the particle pass through the origin?

Solution: We set $x(t) = 0$ and $y(t) = 0$ and solve for t.

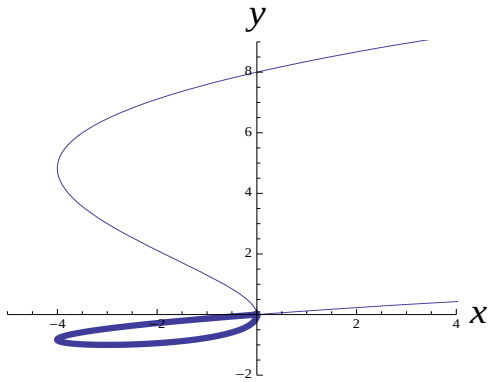
$$x(t) = t^4 - 4t^2 = t^2(t^2 - 4) = t^2(t - 2)(t + 2) = 0$$

gives that $t = -2, 0, 2$, while

$$y(t) = t^2 - 2t = t(t - 2) = 0$$

gives $t = 0, 2$.
Thus, the particle passes through the origin at times $t = 0$ and $t = 2$.

e. [2 points] The graph of the curve traced by these parametric equations is shown below. Find an expression for the length of the closed loop marked in the graph.



Solution: From the given graph and above calculation, we know that the loop is traced out over the time interval $0 \le t \le 2$. The arclength of the loop is given by

$$\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(4t^3 - 8t)^2 + (2t-2)^2}\, dt.$$