3. [11 points] In the late 30th century, Mom’s Friendly Robot Company is the main global robot manufacturing company. The Bending Unit 22 model is designed to contain a backup unit, effectively rendering it immortal. However, a small percentage of the robots suffer a manufacturing defect, in which the backup unit is malfunctional or not present. The function

\[ p(t) = \begin{cases} 
0 & \text{if } t < 0 \\
0.004e^{-t/c} & \text{if } t \geq 0 
\end{cases} \]

gives the probability density for the lifetime of these defective Bending Units 22, where \( c \) is a positive constant and \( t \) is measured in years since the robots are activated. Show all your work to receive full credit.

a. [2 points] Interpret the quantity \( \int_{100}^{140} p(t) \, dt \).

Solution: \( \int_{100}^{140} p(t) \, dt \) gives the fraction of defective Bending Units 22 that have a lifespan between 100 and 140 years.

OR

\( \int_{100}^{140} p(t) \, dt \) gives the probability that a defective Bending Unit 22 will have a lifetime between 100 and 140 years.

b. [4 points] Find the value of \( c \).

Solution: Since \( p(t) \) is a probability density function, we know that \( \int_{-\infty}^{\infty} p(t) \, dt = 1 \). Thus,

\[
1 = \int_{-\infty}^{\infty} p(t) \, dt = \int_{-\infty}^{0} p(t) \, dt + \int_{0}^{\infty} p(t) \, dt = 0 + \int_{0}^{\infty} 0.004e^{-t/c} \, dt
\]

\[
= \lim_{b \to \infty} \int_{0}^{b} 0.004e^{-t/c} \, dt = \lim_{b \to \infty} \left[ -c \cdot 0.004e^{-t/c} \right]_{0}^{b}
\]

\[
= \lim_{b \to \infty} \left( -0.004ce^{-b/c} + 0.004ce^{0} \right) = 0 + 0.004c
\]

\[ 1 = 0.004c \]

\[ c = 250 \]

What is the mean (average) lifespan of a defective Bending Unit 22?

c. [5 points]
Solution: Using $c = 250$ from above and that $p(t) = 0$ for $t < 0$,

$$
\bar{t} = \int_{-\infty}^{\infty} tp(t) \, dt = \int_{-\infty}^{0} tp(t) \, dt + \int_{0}^{\infty} tp(t) \, dt = 0 + \int_{0}^{\infty} 0.004te^{-t/250} \, dt = \lim_{b \to \infty} \int_{0}^{b} 0.004te^{-t/250} \, dt.
$$

Integration by parts with $u = t, dv = 0.004e^{-t/250}$ (and $du = dt, v = -e^{-t/250}$) gives

$$
\bar{t} = \lim_{b \to \infty} \left[ -te^{-t/250} \bigg|_{0}^{b} + \int_{0}^{b} e^{-t/250} \, dt \right] = \lim_{b \to \infty} \left[ -be^{-b/250} + 0 - 250e^{-t/250} \bigg|_{0}^{b} \right]
$$

$$
= \lim_{b \to \infty} \left[ -be^{-b/250} - 250e^{-b/250} + 250e^{0} \right] = 0 - 0 + 250
$$

$$
= 250.
$$

Thus, the mean lifespan of a defective Bending Unit 22 is 250 years.