3. [11 points] In the late 30th century, Mom's Friendly Robot Company is the main global robot manufacturing company. The Bending Unit 22 model is designed to contain a backup unit, effectively rendering it immortal. However, a small percentage of the robots suffer a manufacturing defect, in which the backup unit is malfunctional or not present. The function

$$p(t) = \begin{cases} 0, & \text{if } t < 0 \\ 0.004e^{-t/c}, & \text{if } t \geq 0 \end{cases}$$

gives the probability density for the lifetime of these defective Bending Units 22, where c is a positive constant and t is measured in years since the robots are activated. Show all your work to receive full credit.

a. [2 points] Interpret the quantity $\displaystyle\int_{100}^{140} p(t)\,dt$.

Solution: $\displaystyle\int_{100}^{140} p(t)\,dt$ gives the fraction of defective Bending Units 22 that have a lifespan between 100 and 140 years.

OR

$\displaystyle\int_{100}^{140} p(t)\,dt$ gives the probability that a defective Bending Unit 22 will have a lifetime between 100 and 140 years.

b. [4 points] Find the value of c.

Solution: Since $p(t)$ is a probability density function, we know that $\displaystyle\int_{-\infty}^{\infty} p(t) = 1$. Thus,

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} p(t)\,dt = \int_{-\infty}^{0} p(t)\,dt + \int_{0}^{\infty} p(t)\,dt = 0 + \int_{0}^{\infty} 0.004e^{-t/c}\,dt \\ &= \lim_{b\to\infty} \int_{0}^{b} 0.004e^{-t/c}\,dt = \lim_{b\to\infty} \Big[-c \cdot 0.004e^{-t/c}\Big]_{0}^{b} \\ &= \lim_{b\to\infty} \left(-0.004ce^{-b/c} + 0.004ce^{0}\right) = 0 + 0.004c \\ 1 &= 0.004c \\ c &= 250 \end{aligned}$$

What is the mean (average) lifespan of a defective Bending Unit 22?

c. [5 points]

Solution: Using $c = 250$ from above and that $p(t) = 0$ for $t < 0$,

$$\begin{aligned}\bar{t} &= \int_{-\infty}^{\infty} tp(t)\,dt = \int_{-\infty}^{0} tp(t)\,dt + \int_{0}^{\infty} tp(t)\,dt \\ &= 0 + \int_{0}^{\infty} 0.004te^{-t/250}\,dt = \lim_{b\to\infty}\int_{0}^{b} 0.004te^{-t/250}\,dt.\end{aligned}$$

Integration by parts with $u = t, dv = 0.004e^{-t/250}$(and $du = dt, v = -e^{-t/250}$) gives

$$\begin{aligned}\bar{t} &= \lim_{b\to\infty}\left[-te^{-t/250}\Big|_0^b + \int_0^b e^{-t/250}\,dt\right] = \lim_{b\to\infty}\left[-be^{-b/250} + 0 - 250e^{-t/250}\Big|_0^b\right] \\ &= \lim_{b\to\infty}\left[-be^{-b/250} - 250e^{-b/250} + 250e^{0}\right] = 0 - 0 + 250 \\ &= 250.\end{aligned}$$

Thus, the mean lifespan of a defective Bending Unit 22 is 250 years.