5. [11 points] Determine if the following integrals converge or diverge. If an integral diverges, explain why. If it converges, find the value to which it converges. Mathematical precision is important.

a. [4 points] $\displaystyle\int_{-1}^{2} \frac{1}{\sqrt{2-x}}\, dx$

Solution: The integrand is discontinuous as $x = 2$. Thus, substituting $w = 2 - x, dw = -dx$ we obtain

$$\begin{aligned}
\int_{-1}^{2} \frac{1}{\sqrt{2-x}}\, dx &= \lim_{b\to 2^-} \int_{-1}^{b} (2-x)^{-1/2}\, dx \\
&= \lim_{b\to 2^-} \int_{w(-1)}^{w(b)} w^{-1/2} \cdot -dw \\
&= \lim_{b\to 2^-} -2w^{1/2}\Big|_{w(-1)}^{w(b)} = \lim_{b\to 2^-} -2\sqrt{2-x}\Big|_{-1}^{b} \\
&= \lim_{b\to 2^-} \left(-2\sqrt{2-b} + 2\sqrt{2-(-1)}\right) = 0 + 2\sqrt{3} \\
&= 2\sqrt{3},
\end{aligned}$$

so the integral converges.

b. [4 points] $\displaystyle\int_{10}^{\infty} \frac{5+2\sin(4\theta)}{\theta}\, d\theta$

Solution: Since $-2 \leq 2\sin(4\theta) \leq 2$, we know that $3 \leq 5 + 2\sin(4\theta) \leq 7$, and so that in particular

$$\frac{3}{\theta} \leq \frac{5+2\sin(4\theta)}{\theta}$$

over the interval $[10, \infty)$. Since $\int_{10}^{\infty} \frac{3}{\theta}\, d\theta$ diverges ($p = 1$), we know that $\int_{10}^{\infty} \frac{5+2\sin(4\theta)}{\theta}\, d\theta$ diverges by comparison.

c. [3 points] $\displaystyle\int_{1}^{\infty} \frac{x}{1+x}\, dx$

Solution: Since $\frac{x}{1+x} \to 1$ as $x \to \infty$ (in particular, the integrand does not approach 0), we know that the integral $\int_{1}^{\infty} \frac{x}{1+x}\, dx$ must diverge.
OR

$\frac{1}{2} \leq \frac{x}{x+x} \leq \frac{x}{1+x}$ for $x \geq 1$. Hence $\int_{1}^{\infty} \frac{1}{2} dx \leq \int_{1}^{\infty} \frac{x}{1+x}\, dx$, by the comparison method $\int_{1}^{\infty} \frac{x}{1+x}\, dx$ diverges.