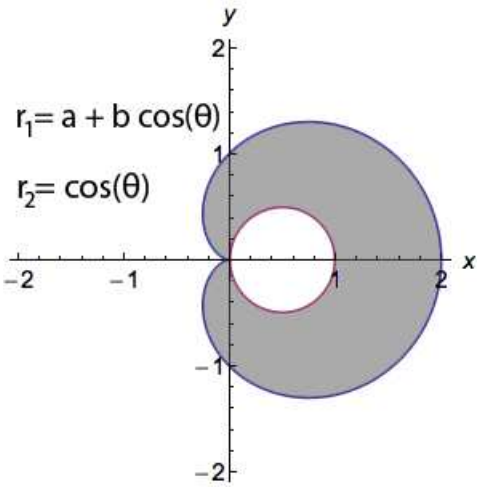
6. [8 points] Members of the recruitment committee for the Mars University (MU) chapter of the fraternity Epsilon Rho Rho are designing a pledge pin to distribute during Rush Week. The pin takes the shape of a cardioid with a circular hole in it. The cardioid is given by a polar equation of the form $r_1 = a + b\cos(\theta)$, while the circular hole has the polar equation $r_2 = \cos(\theta)$. The pin is pictured below, where the x- and y-axes are measured in inches.



a. [5 points] The committee plans on coating one side of the pin in gold plating, which costs 3 dollars per square inch. Give an expression representing the cost to plate one face of the pin in gold. Your answer may involve integrals and the constants a and b.

Solution: We note that the cardioid $r_1 = a + b\cos(\theta)$ is traced out over $0 \le \theta \le 2\pi$, while the circle $r_2 = \cos(\theta)$ is traced out over $0 \le \theta \le \pi$. The area, in square inches, of the pin is then

$$\frac{1}{2}\int_0^{2\pi} (a + b\cos(\theta))^2 \, d\theta - \frac{1}{2}\int_0^{\pi} (\cos(\theta))^2 \, d\theta,$$

so that the total cost to plate one face of the pin is $\frac{3}{2}\left(\int_0^{2\pi}(a + b\cos(\theta))^2\, d\theta - \int_0^{\pi}\cos^2(\theta)\, d\theta\right)$ dollars. One could also note that the inner circle has radius $\frac{1}{2}$, so that its area is just $\pi\left(\frac{1}{2}\right)^2$, and the cost of the gold plating becomes

$$\frac{3}{2}\int_0^{2\pi} (a + b\cos(\theta))^2 \, d\theta - \frac{3}{4}\pi \text{ dollars.}$$

b. [3 points] Find a and b.

Solution: From the given graph, we see that when $\theta = 0, r_1 = 2$ and when $\theta = \frac{\pi}{2}, r_1 = 1$. This yields the system of equations

$$a + b\cos(0) = a + b = 2, \qquad a + b\cos(\frac{\pi}{2}) = a = 1.$$

Thus $a = 1, b = 1$.
Or
from the given graph, we see that when $\theta = 0, r_1 = 0$ and when $\theta = \frac{\pi}{2}, r_1 = -1$. This yields the system of equations

$$a + b\cos(0) = a + b = 0, \qquad a + b\cos(\frac{\pi}{2}) = a = -1.$$

Thus $a = -1, b = 1$.