7. [13 points] A company designs an air filter for a ship's engine room that reduces the amount of fumes in the air by k percent every hour. The machinery in the engine room produces fumes at a rate of 0.02 kilograms per hour. Let $Q(t)$ be the amount in kilograms of fumes in the room t hours after the engines are activated. Hence Q satisfies

$$\frac{dQ}{dt} = 0.02 - \frac{k}{100}Q.$$

a. [9 points] Find a formula for $Q(t)$. Suppose there are no fumes in the air when the engines are activated.

Solution:

$$\begin{aligned} \frac{dQ}{0.02 - \frac{k}{100}Q} &= dt. \\ -\frac{100}{k}\ln\left|0.02 - \frac{k}{100}Q\right| &= t + C. \\ 0.02 - \frac{k}{100}Q &= Ae^{-\frac{k}{100}t} \\ Q(t) &= \frac{100}{k}\left(0.02 - Ae^{-\frac{k}{100}t}\right) \\ Q(t) &= \frac{2}{k}\left(1 - e^{-\frac{k}{100}t}\right) \qquad \text{using} \qquad Q(0) = 0. \end{aligned}$$

b. [2 points] What is the value of $Q(t)$ in the long run?

Solution: $\lim_{t\to\infty} Q(t) = \frac{2}{k}$.

c. [2 points] Air safety regulations require that the *concentration* of fumes in the air not exceed 10^{-4} kilograms per liter at any time. If the volume of air in the engine room is 10^3 liters, for what values of k are the safety regulations met at all times?

Solution: Concentration$=\frac{Q(t)}{\text{Volume}} \leq \frac{\frac{2}{k}}{10^3} \leq 10^{-4}$.

Hence $k \geq 20$.